1. Sketch a graph of the function $f(x) = \begin{cases} -x^2 - 2x + 3 & x < 0 \\ 4 & 0 \le x < 3 \\ x + 1 & x \ge 3 \end{cases}$.

Solution: Before graphing our piecewise function $f(x)$, we create a table of values for each of the three "pieces" of f here. We make note where we have a table entry that will correspond to an open hole on our graph. To more accurately draw our graph, we also note that the first piece of our graph is part of a parabola opening down, the second is a horizontal line at $y = 4$ and the third is part of a line with slope 1.

$-x^2-2x+3$ for $x<0$

x	y
$\vdots$	$\vdots$
-3	0
-2	3
-1	4
0	3

← open hole at $(0,3)$!

4 for $0 \le x < 3$

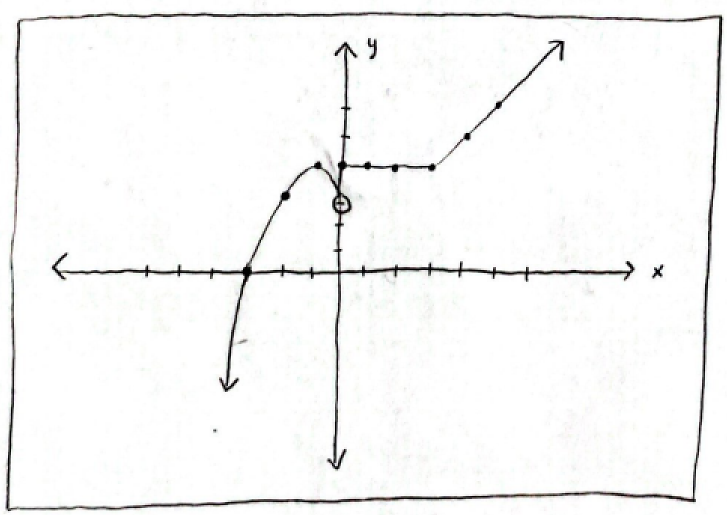
x	y
0	4
1	4
2	4
3	4

← open hole at $(3,4)$!

but, the open hole gets covered by this closed one...

$x+1$ for $x \ge 3$

x	y
3	4
4	5
5	6
$\vdots$	$\vdots$



2. Let $m(x) = \sqrt{x+3}$ and $p(x) = x^2 - 5$.

(a) Find and simplify $\dfrac{(p \circ m)(6)}{(p - m)(1)}$.

(b) Find $m^{-1}(2)$.

Solution:

(a)

$$\begin{aligned}
&\frac{(p \circ m)(6)}{(p - m)(1)} \\
&= \frac{p(m(6))}{p(1) - m(1)} \\
&= \frac{p(\sqrt{6+3})}{1^2 - 5 - \sqrt{1+3}} \\
&= \frac{p(\sqrt{9})}{1 - 5 - \sqrt{4}} \\
&= \frac{p(3)}{1 - 5 - 2} \\
&= \frac{3^2 - 5}{-6} \\
&= \frac{4}{-6} = -\frac{2}{3}
\end{aligned}$$

$$\boxed{\frac{(p \circ m)(6)}{(p - m)(1)} = -\frac{2}{3}}$$

(b) Here, we need to solve for the value of x such that $m(x) = 2$, as shown below.

$$\begin{aligned}
&m(x) = 2 \\
&\Rightarrow 2 = \sqrt{x+3} \\
&\Rightarrow (2)^2 = (\sqrt{x+3})^2 \\
&\Rightarrow 4 = x + 3 \\
&\Rightarrow x = 1
\end{aligned}$$

$$\boxed{m^{-1}(2) = 1}$$

3. The hypotenuse of a right triangle is 3 inches less than twice the base of the triangle. Express the area of the triangle as a function of the base of the triangle.

Solution: Let A represent the area of the triangle, b represent the base of the triangle, and h represent the height of the triangle. We know that $A = \frac{1}{2}bh$, but our goal is to get A in terms of the single variable b. To do this, we need to solve for h in terms of b.

The length of the hypotenuse of the triangle is equal to $2b - 3$. By the Pythagorean Theorem,

$$\begin{aligned}
b^2 + h^2 &= (2b-3)^2 \\
\Rightarrow b^2 + h^2 &= (2b-3)(2b-3) \\
\Rightarrow b^2 + h^2 &= 4b^2 - 6b - 6b + 9 \\
\Rightarrow b^2 + h^2 &= 4b^2 - 12b + 9 \\
\Rightarrow h^2 &= 3b^2 - 12b + 9 \\
\Rightarrow h &= \sqrt{3b^2 - 12b + 9}
\end{aligned}$$

Note that we only took the positive square root in the final step (since the height is a length and hence must be positive).

Therefore,

$$\begin{aligned}
A &= \frac{1}{2}bh \\
&= \frac{1}{2}b\sqrt{3b^2 - 12b + 9}
\end{aligned}$$

$$\boxed{A(b) = \frac{1}{2}b\sqrt{3b^2 - 12b + 9}}$$

4. Find the domain of the function $f(x) = \dfrac{\log_2(x^2 + 3x - 4)}{x - 5}$.

Solution: We have two domain restrictions here. The first restriction is the following:

$$\begin{aligned}
x - 5 &\neq 0 \\
\Rightarrow x &\neq 5
\end{aligned}$$

The second restriction is $x^2 + 3x - 4 > 0$. To unpack this, we need a sign chart for the function $g(x) = x^2 + 3x - 4 = (x+4)(x-1)$. This is shown below, where each of -4 and 1 are zeroes of $g(x)$.

$+$ -4 $-$ 1 $+$

Hence we see that $x^2 + 3x - 4 > 0$ for x in $(-\infty, -4) \cup (1, \infty)$.

Combining the two domain restrictions, we have that x must lie in $(-\infty, -4) \cup (1, \infty)$ and $x \neq 5$. This leads to our final domain $(-\infty, -4) \cup (1, 5) \cup (5, \infty)$.

$\boxed{(-\infty, -4) \cup (1, 5) \cup (5, \infty)}$

5. Given that $2x + 1$ is a factor of the polynomial, find all roots of the equation $2x^3 - 7x^2 + 6x + 5 = 0$. Express any non-real roots in the form $a + bi$.

Solution: We need to be able to factor the polynomial $2x^3 - 7x^2 + 6x + 5$. Since we already know that $2x + 1$ is a factor of $2x^3 - 7x^2 + 6x + 5$, we can use polynomial long division to our advantage, as shown below.

$$
\begin{array}{r|rrrr}
 & x^2 & -4x & +5 & \\
\hline
2x+1 & 2x^3 & -7x^2 & +6x & +5 \\
 & -2x^3 & -x^2 & & \\
\hline
 & & -8x^2 & +6x & \\
 & & 8x^2 & +4x & \\
\hline
 & & & 10x & +5 \\
 & & & -10x & -5 \\
\hline
 & & & & 0
\end{array}
$$

Hence we have $2x^3 - 7x^2 + 6x + 5 = (2x + 1)(x^2 - 4x + 5)$. Now,

$$
\begin{aligned}
2x^3 - 7x^2 + 6x + 5 &= 0 \\
\Rightarrow (2x + 1)(x^2 - 4x + 5) &= 0 \\
\Rightarrow 2x + 1 = 0 \;\text{ or }\; x^2 - 4x + 5 &= 0 \\
\Rightarrow 2x = -1 \;\text{ or }\; x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \\
\Rightarrow x = -\frac{1}{2} \;\text{ or }\; x &= \frac{4 \pm \sqrt{-4}}{2} \\
\Rightarrow x = -\frac{1}{2} \;\text{ or }\; x &= \frac{4 \pm 2i}{2} \\
\Rightarrow x = -\frac{1}{2} \;\text{ or }\; x &= 2 \pm i
\end{aligned}
$$

$\boxed{x = -\frac{1}{2}, 2 - i, 2 + i}$

6. Find the average rate of change of the function $n(x) = \dfrac{1}{x-3}$ from $x = 5$ to $x = 5 + h$. Simplify your answer completely.

Solution:

$$\begin{aligned}
&\frac{n(5+h) - n(5)}{5+h-5} \\
&= \frac{\dfrac{1}{5+h-3} - \dfrac{1}{5-3}}{h} \\
&= \frac{\dfrac{1}{2+h} - \dfrac{1}{2}}{h} \\
&= \frac{\dfrac{1}{2+h} \cdot \dfrac{2}{2} - \dfrac{1}{2} \cdot \dfrac{2+h}{2+h}}{h} \\
&= \frac{\dfrac{2}{2(2+h)} - \dfrac{2+h}{2(2+h)}}{h} \\
&= \frac{\dfrac{2-(2+h)}{2(2+h)}}{h} \\
&= \frac{2-(2+h)}{2(2+h)} \cdot \frac{1}{h} \\
&= \frac{2-(2+h)}{2h(2+h)} \\
&= \frac{2-2-h}{2h(2+h)} \\
&= \frac{-h}{2h(2+h)} \\
&= \frac{-1}{2(2+h)}
\end{aligned}$$

$$\boxed{\frac{-1}{2(2+h)}}$$

7. Graph the function $g(x) = \dfrac{x+1}{(x-4)(x+3)^2}$, labeling all intercepts and asymptotes.

Solution: First, we set the denominator of $g(x)$ equal to zero to solve for the vertical asymptotes:

$$(x-4)(x+3)^2 = 0$$

$$
\begin{aligned}
\Rightarrow x-4=0 \;\text{ or }\; (x+3)^2=0\\
\Rightarrow x=4 \;\text{ or }\; (x+3)(x+3)=0\\
\Rightarrow x=4 \;\text{ or }\; x+3=0 \;\text{ or }\; x+3=0\\
\Rightarrow x=4 \;\text{ or }\; x=-3
\end{aligned}
$$

Hence we have two vertical asymptotes at $x = 4$ and $x = -3$.

Next, since the degree of the denominator of $g(x)$ (which is 3) is greater than the degree of the numerator of $g(x)$ (which is 1), we have a horizontal asymptote at $y = 0$.

Let's now solve for the zeroes of $g(x)$ below:

$$
\begin{aligned}
g(x)=0\\
\Rightarrow \frac{x+1}{(x-4)(x+3)^2}=0\\
\Rightarrow \frac{x+1}{(x-4)(x+3)^2}\cdot(x-4)(x+3)^2=0\cdot(x-4)(x+3)^2\\
\Rightarrow x+1=0\\
\Rightarrow x=-1
\end{aligned}
$$

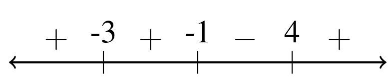
Hence g(x) has a single zero at $x = -1$. This also means that $g(x)$ has a single x-intercept at $(-1, 0)$.

To find the y-intercept of $g(x)$, we find $g(0)$ below:

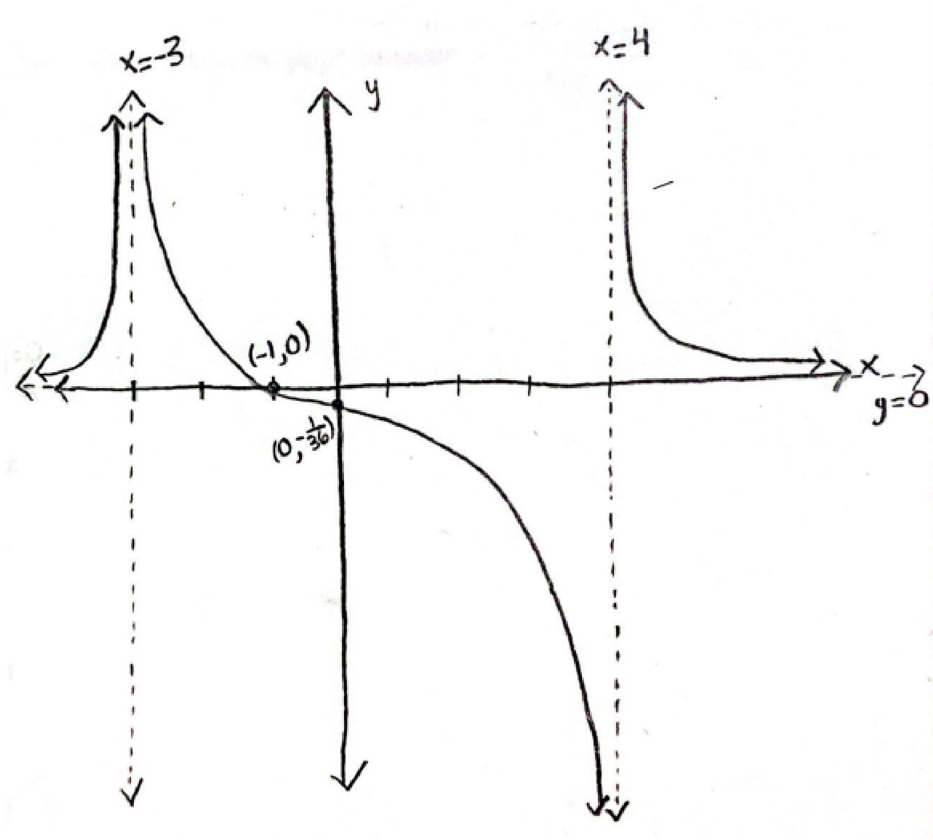
$$
\begin{aligned}
g(0)\\
=\frac{0+1}{(0-4)(0+3)^2}\\
=\frac{1}{(-4)(9)}\\
=-\frac{1}{36}
\end{aligned}
$$

Hence $g(x)$ has a y-intercept at $\left(0, -\frac{1}{36}\right)$.

Lastly, we create a sign chart for $g(x)$ below, with key points at the vertical asymptotes and zeroes from earlier.



The final graph is shown below.



8. A local coffee shop wants to produce coffee cups. The shop has determined that when x coffee cups are made, the cost per coffee cup is determined by $C(x) = \frac{1}{2}x^2 - 8x + 68$.

(a) What is the minimum cost?

(b) How many coffee cups should be produced to yield the minimum cost?

Solution:

(a) The graph of $C(x)$ is a parabola opening up. The x-coordinate of the vertex of the parabola is given by $\dfrac{-(-8)}{2\left(\frac{1}{2}\right)} = \dfrac{8}{1} = 8$. This represents the number of coffee cups made such that the cost per coffee cup is lowest. The y-coordinate of the vertex of the parabola is then given by $C(8) = \frac{1}{2}(8)^2 - 8(8) + 68 = \frac{1}{2}(64) - 64 + 68 = 32 - 64 + 68 = 36$. This represents the lowest cost per coffee cup.

$\boxed{\text{36 dollars}}$

(b) $\boxed{8 \text{ coffee cups}}$

9. Simplify each expression completely.

(a) $\log_4(\sqrt{8})$

(b) $49^{\log_7(3)+2\log_7(2)}$

Solution:

(a)

$$\begin{aligned} &\log_4(\sqrt{8}) \\ &= \log_4(8^{\frac{1}{2}}) \\ &= \frac{1}{2}\log_4(8) \\ &= \frac{1}{2}\log_4(2^3) \\ &= \frac{1}{2}\cdot 3\cdot \log_4(2) \\ &= \frac{1}{2}\cdot 3\cdot \frac{1}{2} \\ &= \frac{3}{4} \end{aligned}$$

$$\boxed{\log_4(\sqrt{8}) = \frac{3}{4}}$$

(b)

$$\begin{aligned} &49^{\log_7(3)+2\log_7(2)} \\ &= 49^{\log_7(3)+\log_7(2^2)} \\ &= 49^{\log_7(3)+\log_7(4)} \\ &= 49^{\log_7(3\cdot 4)} \\ &= 49^{\log_7(12)} \\ &= (7^2)^{\log_7(12)} \\ &= 7^{2\log_7(12)} \\ &= 7^{\log_7(12^2)} \\ &= 12^2 = 144 \end{aligned}$$

$$\boxed{49^{\log_7(3)+2\log_7(2)} = 144}$$

10. A bacteria culture decays exponentially according to the function $Q(t) = Q_0 e^{rt}$, If the culture decays from 140 grams to 20 grams in 5 hours, find the time it takes for the population to decrease to half its initial size. Simplify your answer completely.

Solution: We know that $Q_0 = 140$, so the exponential decay function is of the form $Q(t) = 140e^{rt}$. We also know that $Q(5) = 20$, and we can use this to solve for r as follows:

$$\begin{aligned} Q(5) &= 20 \\ \Rightarrow 20 &= 140e^{5r} \\ \Rightarrow \frac{20}{140} &= e^{5r} \\ \Rightarrow \frac{1}{7} &= e^{5r} \\ \Rightarrow \ln\left(\frac{1}{7}\right) &= \ln\left(e^{5r}\right) \\ \Rightarrow \ln\left(\frac{1}{7}\right) &= 5r \\ \Rightarrow r &= \frac{\ln\left(\frac{1}{7}\right)}{5} \end{aligned}$$

Therefore, the exponential decay function is of the form $Q(t) = 140e^{\frac{\ln\left(\frac{1}{7}\right)}{5}t}$. Since half of 140 is equal to 70, we want to find the value of t such that $Q(t) = 70$. The work for this is as follows:

$$\begin{aligned} Q(t) &= 70 \\ \Rightarrow 70 &= 140e^{\frac{\ln\left(\frac{1}{7}\right)}{5}t} \\ \Rightarrow \frac{70}{140} &= e^{\frac{\ln\left(\frac{1}{7}\right)}{5}t} \\ \Rightarrow \frac{1}{2} &= e^{\frac{\ln\left(\frac{1}{7}\right)}{5}t} \\ \Rightarrow \ln\left(\frac{1}{2}\right) &= \ln\left(e^{\frac{\ln\left(\frac{1}{7}\right)}{5}t}\right) \\ \Rightarrow \ln\left(\frac{1}{2}\right) &= \frac{\ln\left(\frac{1}{7}\right)}{5}t \\ \Rightarrow \frac{5}{\ln\left(\frac{1}{7}\right)} \cdot \ln\left(\frac{1}{2}\right) &= \frac{5}{\ln\left(\frac{1}{7}\right)} \cdot \frac{\ln\left(\frac{1}{7}\right)}{5}t \\ \Rightarrow t &= \frac{5\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{7}\right)} \end{aligned}$$

$$\boxed{\frac{5\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{7}\right)} \text{ hours}}$$

11. Graph $h(x) = -\log_2(x+4)+3$. Label all intercepts and asymptotes.

Solution: We will graph $h(x) = -\log_2(x+4)+3$ by performing transformations to the graph of $\log_2(x)$. This "parent graph" has a vertical asymptote at $x = 0$ and passes through the points $(1, 0)$ and $(2, 1)$. We will track the asymptote and these two points with each transformation. Keep in mind the proper order of transformations: horizontal shift, reflection over the y-axis, reflection over the x-axis, and vertical shift. The fourth and final graph in the image below is our answer here.

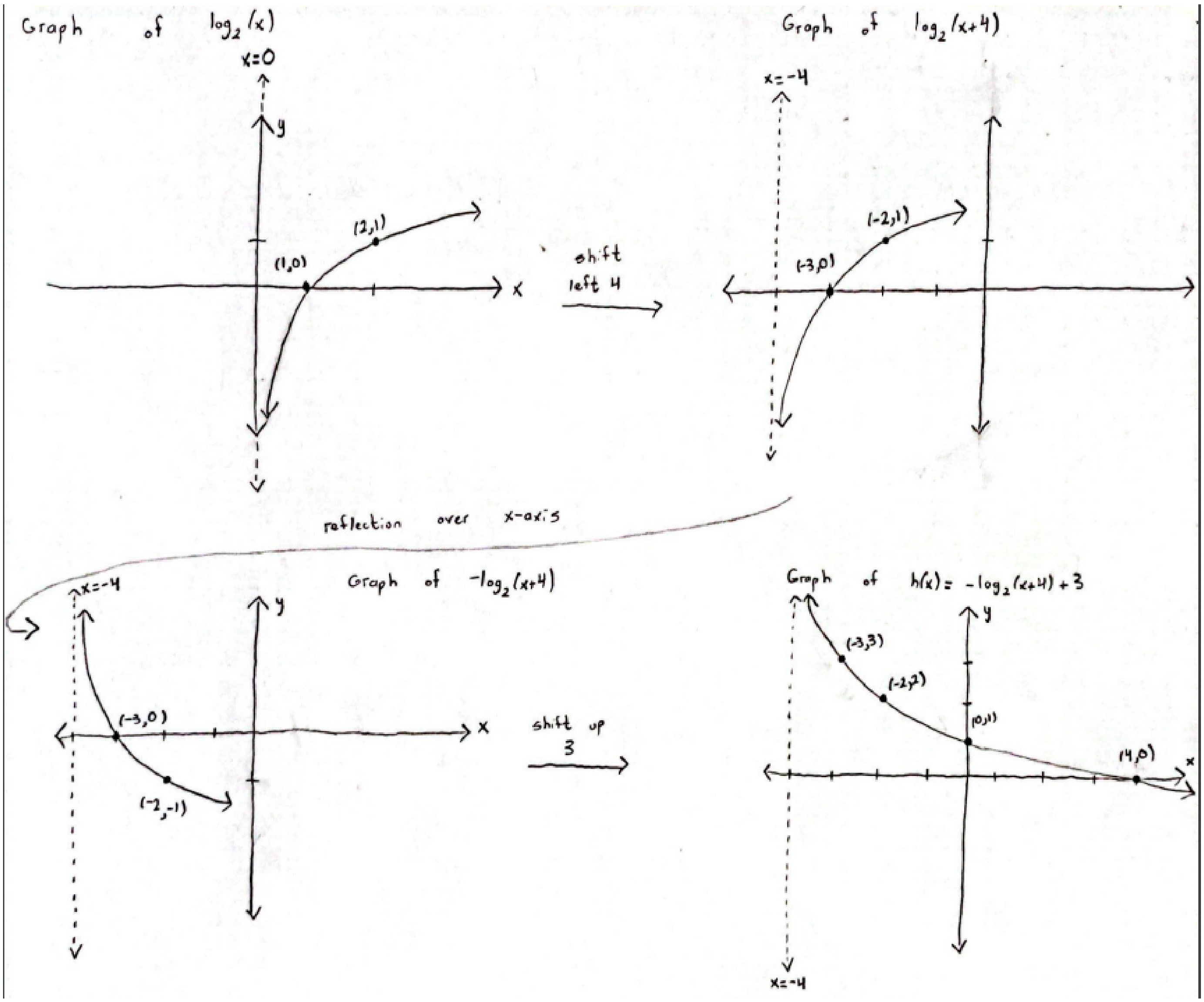
We also need to find the intercepts of $h(x) = -\log_2(x+4)+3$. To get the y-intercept, we find $h(0)$:

$$\begin{aligned} & h(0) \\ = & -\log_2(0+4)+3 \\ = & -\log_2(4)+3 \\ = & -2+3 = 1 \end{aligned}$$

Hence $h(x)$ has a y-intercept at $(0, 1)$. To get the x-intercept, we set $h(x) = 0$ and solve for x:

$$\begin{aligned} 0 &= -\log_2(x+4)+3 \\ \Rightarrow -3 &= -\log_2(x+4) \\ \Rightarrow 3 &= \log_2(x+4) \\ \Rightarrow 2^3 &= x+4 \\ \Rightarrow 8 &= x+4 \\ \Rightarrow x &= 4 \end{aligned}$$

Hence $h(x)$ has an x-intercept at $(4, 0)$.



12. Evaluate each of the following.

a) $\sec\left(\frac{-17\pi}{6}\right)$ b) $\sin^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$

Solution:

(a)

$$\sec\left(\frac{-17\pi}{6}\right)$$

$$
\begin{aligned}
&= \frac{1}{\cos\left(\dfrac{-17\pi}{6}\right)} \\
&= \frac{1}{\cos\left(\dfrac{-17\pi}{6} + 2\pi\right)} \\
&= \frac{1}{\cos\left(\dfrac{-17\pi}{6} + \dfrac{12\pi}{6}\right)} \\
&= \frac{1}{\cos\left(\dfrac{-5\pi}{6}\right)} \\
&= \frac{1}{\cos\left(\dfrac{-5\pi}{6} + 2\pi\right)} \\
&= \frac{1}{\cos\left(\dfrac{-5\pi}{6} + \dfrac{12\pi}{6}\right)} \\
&= \frac{1}{\cos\left(\dfrac{7\pi}{6}\right)} \\
&= \frac{1}{\dfrac{-\sqrt{3}}{2}} \\
&= \frac{1}{1} \cdot -\frac{2}{\sqrt{3}} \\
&= -\frac{2}{\sqrt{3}} \\
&= -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{-2\sqrt{3}}{3}
\end{aligned}
$$

$$\boxed{\sec\left(\frac{-17\pi}{6}\right) = \frac{-2\sqrt{3}}{3}}$$

(b) The range of $\sin^{-1}(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. With this in mind, we have the following:

$$\sin^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

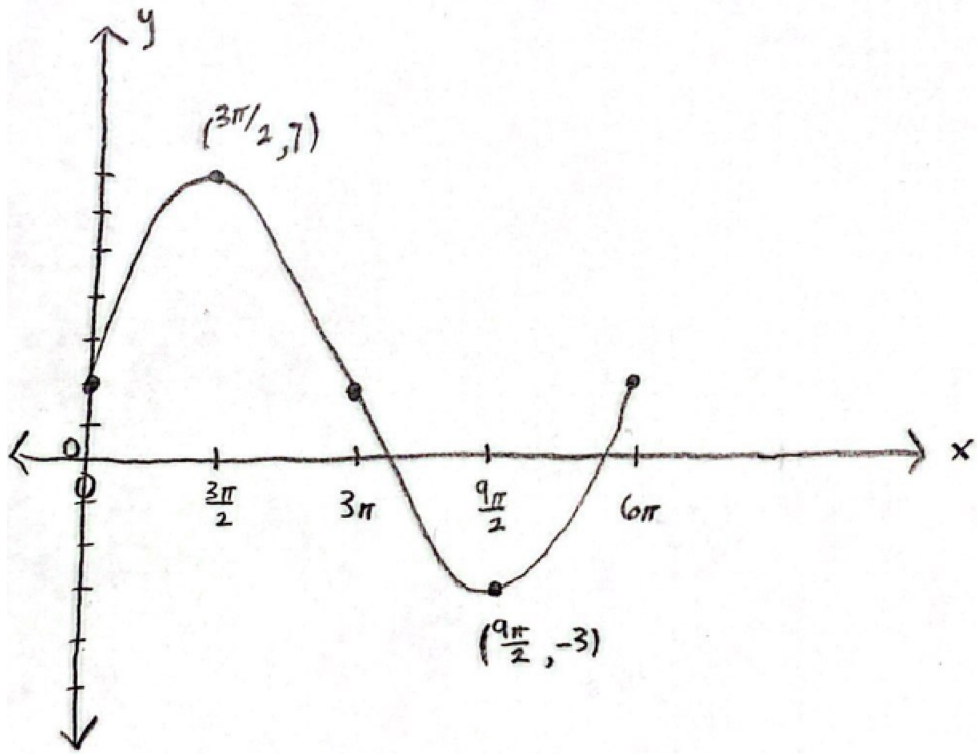
$$\boxed{\sin^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right) = -\frac{\pi}{4}}$$

13. Graph one complete period of the function $g(x) = 5\sin\left(\frac{1}{3}x\right) + 2$, labeling the highest and lowest points.

Solution: It is helpful to create a table here. If the value of $\frac{1}{3}x$ inside of the parentheses were equal to any of the 5 major angles on the unit circle (0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π), we would immediately know $\sin\left(\frac{1}{3}x\right)$. Therefore, in the table below, we fill in those 5 major angles (in the second column) for $\frac{1}{3}x$ first, then fill in the first column for the corresponding values of x that will be on our graph, and then easily fill in the third and fourth columns from our knowledge of the unit circle.

x	$\frac{1}{3}x$	$\sin\left(\frac{1}{3}x\right)$	$5\sin\left(\frac{1}{3}x\right)+2$
0	0	0	2
$\frac{3\pi}{2}$	$\frac{\pi}{2}$	1	7
3π	π	0	2
$\frac{9\pi}{2}$	$\frac{3\pi}{2}$	-1	-3
6π	2π	0	2

The graph is shown below.

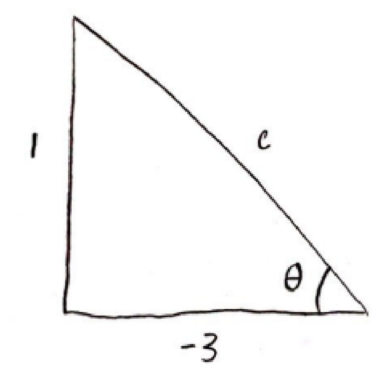


14. Given that $\cot(\theta) = -3$ and $\cos(\theta) < 0$, find $\sin\left(\theta - \dfrac{5\pi}{3}\right)$. Simplify your answer completely.

Solution: By use of the formula $\sin(x - y) = \sin(x)\cos(y) - \sin(y)\cos(x)$, we have the following:

$$\begin{aligned} &\sin\left(\theta - \frac{5\pi}{3}\right) \\ &= \sin(\theta)\cos\left(\frac{5\pi}{3}\right) - \sin\left(\frac{5\pi}{3}\right)\cos(\theta) \\ &= \sin(\theta)\cdot\frac{1}{2} - \left(-\frac{\sqrt{3}}{2}\right)\cos(\theta) \\ &= \frac{1}{2}\sin(\theta) + \frac{\sqrt{3}}{2}\cos(\theta) \end{aligned}$$

Hence we still need to find $\sin(\theta)$ and $\cos(\theta)$. We know that $\tan(\theta) = \dfrac{1}{\cot(\theta)} = -\dfrac{1}{3}$ and $\cos(\theta) < 0$. Therefore, since $\tan(\theta)$ and $\cos(\theta)$ are both negative, θ must lie in Quadrant II. We now draw the appropriate right triangle with θ in Quadrant II and $\tan(\theta) = -\dfrac{1}{3}$ below, keeping in mind that $\tan(\theta)$ represents the side length opposite from θ divided by the side length adjacent to θ.



In order to determine both $\sin(\theta)$, which represents the side length opposite from θ divided by the hypotenuse length, and $\cos(\theta)$, which represents the side length adjacent to θ divided by the hypotenuse length, we need to use the Pythagorean Theorem to find the hypotenuse length c. This work is shown below.

$$\begin{aligned} 1^2 + (-3)^2 &= c^2 \\ 1 + 9 &= c^2 \\ 10 &= c^2 \\ c &= \sqrt{10} \end{aligned}$$

Note that we've only taken the positive square root in the final step; the hypotenuse length is always taken to be positive in these types of problems.

Therefore, we have the following:

$$\begin{aligned} &\sin\left(\theta - \frac{5\pi}{3}\right) \\ &= \frac{1}{2}\sin(\theta) + \frac{\sqrt{3}}{2}\cos(\theta) \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{10}} + \frac{\sqrt{3}}{2} \cdot \frac{-3}{\sqrt{10}} \\ &= \frac{1}{2\sqrt{10}} - \frac{3\sqrt{3}}{2\sqrt{10}} \\ &= \frac{1 - 3\sqrt{3}}{2\sqrt{10}} \end{aligned}$$

$$\boxed{\sin\left(\theta - \frac{5\pi}{3}\right) = \frac{1 - 3\sqrt{3}}{2\sqrt{10}}}$$

15. Find all primary solutions ($0 \leq \theta < 2\pi$) of the trigonometric equation $2\cos^2(\theta) = 9\cos(\theta) + 5$.

Solution:

$$\begin{aligned}
2\cos^2(\theta) &= 9\cos(\theta) + 5 \\
\Rightarrow 2\cos^2(\theta) - 9\cos(\theta) - 5 &= 0 \\
\Rightarrow (2\cos(\theta) + 1)(\cos(\theta) - 5) &= 0 \\
2\cos(\theta) + 1 = 0 \text{ or } \cos(\theta) - 5 &= 0 \\
\Rightarrow 2\cos(\theta) = -1 \text{ or } \underbrace{\cos(\theta) = 5}_{\text{No solution}} \\
\Rightarrow \cos(\theta) &= -\frac{1}{2} \\
\Rightarrow \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}
\end{aligned}$$

$$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$$

16. Verify the identity: $\dfrac{\sin(2x)}{1 + \cos(2x)} = \tan(x)$.

Solution:

$$\begin{aligned}
&\frac{\sin(2x)}{1 + \cos(2x)} \\
&= \frac{2\sin(x)\cos(x)}{1 + \cos^2(x) - \sin^2(x)} \\
&= \frac{2\sin(x)\cos(x)}{\cos^2(x) + 1 - \sin^2(x)} \\
&= \frac{2\sin(x)\cos(x)}{\cos^2(x) + \cos^2(x)} \\
&= \frac{2\sin(x)\cos(x)}{2\cos^2(x)} \\
&= \frac{\sin(x)\cos(x)}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos(x)} \\
&= \tan(x)
\end{aligned}$$