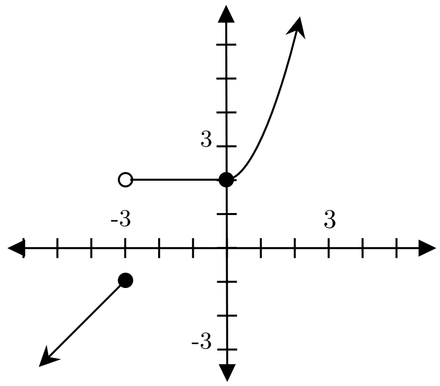
MAT 1800 Winter '23 Final Exam Solutions

1. Use a table to find the values of function in it's respective domain and checking the endpoints.



x	$y = x + 1$
-3	-2
-4	-3

x	$y = 2$
-3	2
-2	2
0	2

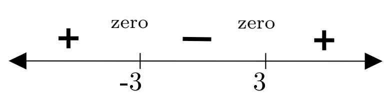
x	$y = x^2 + 2$
0	2
1	3
2	6

2. Solve for x values which make the denominator zero:

   $\log_2(x^2 - 9) - 4 = 0 \implies \log_2(x^2 - 9) = 4 \implies 2^{\log_2(x^2-9)} = 2^4 \implies x^2 - 9 = 16 \implies x^2 = 25$ $\implies x = 5, -5$. We therefore exclude 5 and -5 from the domain of $f(x)$.

   Solve for the domain of $\log_2(x^2 - 9)$:

   Logarithms require positive arguments to be defined, so let's make a sign chart for $x^2 - 9$. The zero's of $x^2 - 9$ are 3 and -3. Pick test values to determine the sign. When $x = 0$, $0^2 - 9 = -9$. When $x = -4$, $(-4)^2 - 9 = 7$. When $x = 4$, $4^2 - 9 = 7$. So, the sign chart looks like this:



   Thus, the domain of $\log_2(x^2-9)$ is $(-\infty, -3) \cup (3, \infty)$. Since there are no other domain issues with $f(x)$, we can put together our information to get the domain for $f(x)$: $(-\infty, -5) \cup (-5, -3) \cup (3, 5) \cup (5, \infty)$.

3. (a) First, we have $g(4) = \frac{3 \cdot 4}{4-2} = \frac{12}{2} = 6$, so

   $(f \circ g)(4) = f(g(4)) = f(6) = \sqrt{6-5} = 1$. Putting these together we have $\frac{(f \circ g)(4)}{g(4)} = \frac{1}{6}$.

   (b) $g^{-1}\left(\frac{9}{4}\right)$ is asking for what values of x is $g(x) = \frac{9}{4}$.

   [come back too. need to formulate a better way to do this]

4. If we call the length of one side of the square base w, then we get the equation $w^2 = 100$. So, $w = 10$. To find the surface area of the box, add the area of each side: $SA = 2w^2 + 4wh$. (SA for surface area). Since we solved for w, substitute it into the surface area equation to get the final equation $SA = 200 + 40h$.

5. The height of the ball is modeled by a (downward opening) parabola. The maximum height will correspond to the y-coordinate of the vertex. To find this we must complete the square and put the equation into vertex form:

   $h(t) = -\frac{1}{8}t^2 + 3t \implies h(t) = -\frac{1}{8}\left(t^2 - 24t\right) \implies h(t) = -\frac{1}{8}\left(t^2 - 24t + 144\right) - 144\left(-\frac{1}{8}\right)$

   $\implies h(t) = -\frac{1}{8}\left(t - 12\right)^2 + 18$

   The vertex of the parabola $(12, 18)$, so in particular the maximum height the ball reaches is 18 feet.

6. Since -3 is root, $(x-3)$ is a factor of the polynomial. Use polynomial long division to find the other factor:

$$
\begin{array}{r|l}
 & \quad 2x^2+2x+3 \\
\hline
x+3 & 2x^3+8x^2+9x+9 \\
 & \underline{-(2x^3+6x^2)} \\
 & \qquad 2x^2+9x \\
 & \quad \underline{-(2x^2+6x)} \\
 & \qquad\qquad 3x+9 \\
 & \qquad\quad \underline{-(3x+9)} \\
 & \qquad\qquad\quad 0 \ \checkmark
\end{array}
$$

This tells us that $2x^3+8x^2+9x+9=(x-3)(2x^2+2x+3)$. We can use the quadratic formula to find the remaining roots of $2x^2+2x+3$:

$$\frac{-2\pm\sqrt{4-4(2)(3)}}{2(2)}=\frac{-2\pm\sqrt{-20}}{4}=\frac{-2\pm 2i\sqrt{5}}{4}=-\frac{1}{2}\pm\frac{\sqrt{5}}{2}i$$

7. Use the average rate of change formula (the slope formula) to find:

$$\frac{f(2+h)-f(2)}{2+h-2}=\frac{f(2+h)-f(2)}{h}=\frac{3(2+h)^2+2(2+h)-[3(2)^2+2(2)]}{h}$$

$$\frac{3(4+4h+h^2)+4+2h-[12+4]}{h}=\frac{12+12h+3h^2+4+2h-16}{h}=\frac{14h+3h^2}{h}=14+3h$$

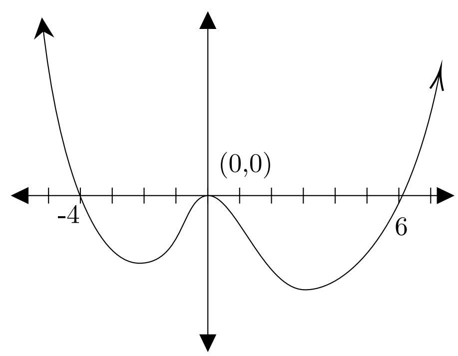
8. The y-intercept is the point when $x=0$. Solve: $f(0)=0^4-2\cdot 0^3-24\cdot 0^2=0$. So, the y-intercept is at $(0,0)$. To find the x-intercept(s), we need to solve the equation $f(x)=0$:

$$f(x)=x^4-2x^3-24x^2=x^2(x^2-2x-24)=x^2(x-6)(x+4)=0$$

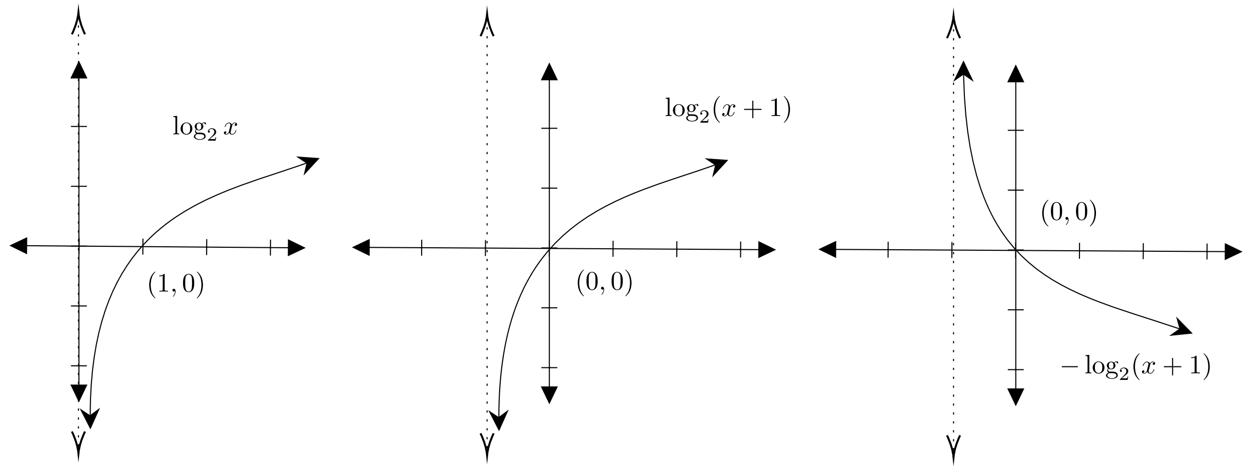
Then we can see that the x-intercepts of $f(x)$ are at $x=0, x=6$, and $x=-4$ on the x-axis. Since this function is a polynomial, it does not have any vertical or horizontal asymptotes. To sketch the graph we need a sign chart to determine the sign of the function and then we use the zero's we found earlier.

Sign Chart:

+	Zero	−	Zero	−	Zero	+
	-4		0		6	



9. Let's graph this is by using transformations. The graph of $-\log_2(x+1)$ can be obtained by starting with the graph of $\log_2(x)$, then shifting the graph one unit to the left to obtain $\log_2(x+1)$, and finally reflecting over the x-axis to obtain $-\log_2(x+1)$:



10. a)

$$\log_4\left(\sqrt{8}\right) = \log_4\left(8^{\frac{1}{2}}\right) = \frac{1}{2}\log_4(8) = \frac{1}{2}\log_4\left(2^3\right) = \frac{3}{2}\log_4(2) = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$$

b)

$$e^{\ln(10)+\frac{1}{2}\ln(4)} = e^{\ln(10)+\ln\left(4^{\frac{1}{2}}\right)} = e^{\ln(10)+\ln\left(\sqrt{4}\right)} = e^{\ln(10)+\ln(2)} = e^{\ln(10\cdot 2)} = e^{\ln(20)} = 20$$

11. The initial amount of leaves is A_0 is 40. First, to find the rate r, we plug in the point $(2, 60)$ into the equation and solve for r. (That is, at $t = 2$ we have $A(2) = 60$:

$$60 = 40e^{2r} \implies \frac{6}{4} = e^2 r \implies \ln\left(\frac{3}{2}\right) = \ln\left(e^{2r}\right) \implies \ln\left(\frac{3}{2}\right) = 2r \implies \frac{1}{2}\ln\left(\frac{3}{2}\right) = r$$

Now we can substitute our known values into the equation and find $A(6)$:

$$A(6) = 40e^{\frac{1}{2}\ln\left(\frac{3}{2}\right)\cdot 6} = 40e^{3\ln\left(\frac{3}{2}\right)\cdot} = 40e^{\ln\left(\left(\frac{3}{2}\right)^3\right)\cdot} = 40\left(\frac{3}{2}\right)^3 = 40\left(\frac{27}{8}\right) = 135 \text{ leaves}$$

12. For these questions you will need to refer to the unit circle.

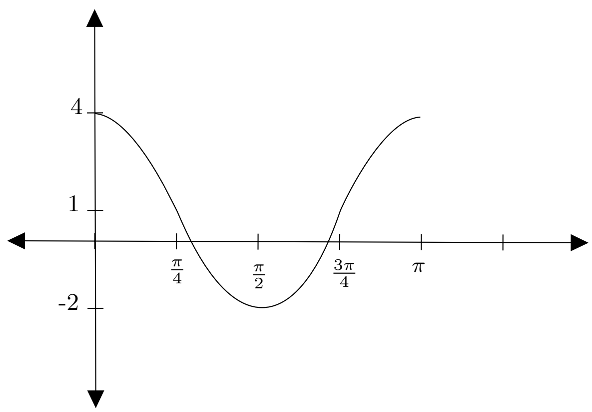
a) $\sec\left(\frac{11\pi}{4}\right) = \frac{1}{\cos\left(\frac{11\pi}{4}\right)} = \frac{1}{\cos\left(\frac{11\pi}{4}-2\pi\right)} = \frac{1}{\cos\left(\frac{3\pi}{4}\right)} = \frac{1}{\frac{-\sqrt{2}}{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

b) $\tan\left(-\frac{2\pi}{3}\right) = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\frac{\sqrt{3}}{2}\cdot -\frac{2}{1} = \sqrt{3}$

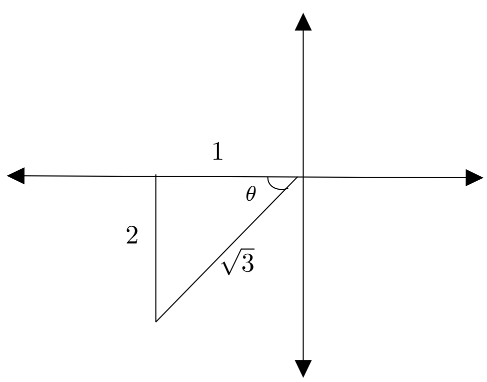
c) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

13. a) For a cosine function $f(x) = a\cos(bx) + c$, the period is $\frac{2\pi}{|b|}$ and the amplitude is $|a|$. For $f(x) = 3\cos(2x) + 1$, the period is π and the amplitude is 3.

b) Start with the base graph for cosine. It's amplitude is 3 so it should reach 3 and -3 as it's maximum and minimum. Then, since it's shifted up one, the graph's new maximum and minimum values are 4 and -2. Lastly note that since the period is π, the graph should go from 0 to π along the x-axis and splitting it evenly into four sections will help graph as well.



14. For this problem we need to set up a triangle. Since $\cot(\theta)$ is positive (i.e. $\tan(\theta)$ is positive), and $cos(\theta)$ is negative, we can determine that our triangle is in the third quadrant. Since, $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{2}$, our triangle should have opposite side length 2 and adjacent side length 1. The last side length is found by using the Pythagorean Theorem: $1^2 + 2^2 = c^2 \implies 5 = c^2 \implies c = \sqrt{5}$. The picture then looks like this:



Using the double angle formula for sine we get $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. Now using our picture we can easily find $\sin(\theta)$ and $\cos(\theta)$: $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{\sqrt{5}}$ and $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{5}}$. So, $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = \frac{4}{5}$.

15.

$$2\sin^2(\theta) + \sin(\theta) - 1 = 0 \implies (2\sin(\theta) - 1)(\sin(\theta) + 1) = 0 \implies 2\sin(\theta) - 1 = 0 \ or \ sin(\theta) + 1 = 0$$

$$\text{When } 2\sin(\theta) - 1 = 0 \implies \sin\theta = \frac{1}{2} \implies \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{When } \sin(\theta) + 1 = 0 \implies \sin(\theta) = -1 \implies \theta = \frac{3\pi}{2}$$

So the final answer is all of the angles we have $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

16. To solve, keep one side fixed and try to manipulate the other side algebraically and with trig identities to get the fixed side. For this problem we will keep the right hand side fixed.

$$\sec(x) - \cos(x) = \tan(x) \cdot \sin(x)$$

$$\frac{1}{\cos(x)} - \cos(x) =$$

$$\frac{1}{\cos(x)} - \frac{\cos^2(x)}{\cos(x)} =$$

$$\frac{1 - \cos^2(x)}{\cos(x)} =$$

$$\frac{\sin^2(x)}{\cos(x)} =$$

$$\frac{\sin(x)\cos(x)}{\sin(x)} =$$

$$\tan(x) \cdot \sin(x)$$