MAT 2010 Fall'20

1. Apply the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)^2}{3+(x+h)} - \dfrac{x^2}{3+x}}{h} = \lim_{h\to 0} \frac{1}{h}\Big(\frac{(x+h)^2(3+x) - x^2(3+x+h)}{(3+x+h)(3+x)}\Big) =$$

$$\lim_{h\to 0} \frac{1}{h}\Big(\frac{(x^2+2xh+h^2)(3+x) - 3x^2 - x^3 - x^2h}{(3+x+h)(3+x)}\Big) = \lim_{h\to 0} \frac{1}{h}\Big(\frac{x^3+(3+2h)x^2+(6h+h^2)x+3h^2-3x^2-x^3-x^2h}{(3+x+h)(3+x)}\Big) =$$

$$\lim_{h\to 0} \frac{1}{h}\Big(\frac{hx^2+(6h+h^2)x+3h^2}{(3+x+h)(3+x)}\Big) = \lim_{h\to 0} \frac{x^2+(6+h)+3h}{(3+x+h)(3+x)} = \boxed{\frac{x^2+6x}{(3+x)^2}}$$

2. (a) Solution 1:

$$\lim_{t\to 0} \frac{\sqrt{9+t}-3}{7} = \lim_{t\to 0} \frac{t}{t(\sqrt{9+t}+3)} = \lim_{t\to 0} \frac{1}{\sqrt{9+t}+3} = \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

Note: we multiply both numerator and denominator by the inverse $\sqrt{9+t}+3$:

$$\frac{\sqrt{9+t}-3}{t} . \frac{\sqrt{9+t}+3}{\sqrt{9+t}+3} = \frac{9+t-9}{t(\sqrt{9+t}+3)} = \frac{1}{\sqrt{9+t}+3}$$

Solution 2:(preferred solution)

By L'Hospital's Rule:

$$\lim_{t\to 0} \frac{1}{2}(9+t)^{-\frac{1}{2}} = \lim_{t\to 0} \frac{1}{2(\sqrt{9+t})} = \boxed{\frac{1}{6}}$$

(b) $\displaystyle\lim_{x\to -2} \frac{3x^3-12x}{x^2-3x-10} = „\frac{0}{0}"$ Indeterminate form

By l'Hospital's Rule:

$$\lim_{x\to -2} \frac{9x^2-12}{2x-3} = \frac{9(-2)^2-12}{2(-2)-3} = \frac{24}{-7} = \boxed{\frac{-24}{7}}$$

3. (a) $f(x) = x^{\frac{4}{3}}.\sec(x)$

   Apply product rule: $f'(x) = (x^{\frac{4}{3}}.\sec(x))' = \frac{4}{3}x^{\frac{1}{3}}.\sec(x) + x^{\frac{4}{3}}.(\sec(x)\tan(x)) = \boxed{x^{\frac{1}{3}}\sec(x)[\frac{4}{3} + x\tan(x)]}$

   (b) $g(x) = \frac{\tan(x)}{x^3+1}$

   Apply quotient rule: $g'(x) = (\frac{\tan(x)}{x^3+1})' = (\frac{(x^3+1))(\sec^2(x)) - (\tan(x))(3x^2)}{(x^3+1)^2} = \boxed{\frac{\sec^2(x)}{x^3+1} - \frac{3x^2\tan(x)}{(x^3+1)^2}}$

   (c) $h(x) = \arcsin(5x) = \sin^{-1}(5x)$ (Note that: $(sin^{-1}(u))' = \frac{u'}{\sqrt{1-u^2}}$)

   Apply Chain Rule: $h'(x) = \frac{1}{\sqrt{1-(5x)^2}}.5 = \boxed{\frac{5}{\sqrt{1-25x^2}}}$

4. (a) $\int \left[\frac{5x+3-x\sec^2(x)}{x}\right]dx = \int \left(5 + \frac{3}{x} - \sec^2(x)\right)dx = \int 5dx + 3\int \frac{1}{x}dx - \int \sec^2(x)dx =$

   $\boxed{5x + 3\ln|x| - \tan(x) + C}$

   (b) $\int_0^{\frac{\pi}{2}} \left[e^x - \cos(x)\right]dx = \left(e^x - \sin(x)\right)\Big|_0^{\frac{\pi}{2}} = (e^{\frac{\pi}{2}} - \sin(\frac{\pi}{2})) - (e^0 - \sin(0)) = e^{\frac{\pi}{2}} - 1 - 1 =$

   $\boxed{e^{\frac{\pi}{2}} - 2}$

5. Let $y = f(x)$ so $\sin(x.f(x)) + (f(x))^3 = x^2$

   Differentiate:

   $$\frac{d}{dx}\left(\sin(x.f(x)) + (f(x))^3\right) = \frac{d}{dx}(x^2) \Longrightarrow \cos(x.f(x))(1.f(x) + x.f'(x)) + 3(f(x))^2.f'(x) = 2x$$

   $$\Longrightarrow \cos(xy)(y + x\frac{dy}{dx}) + 3y^2.\frac{dy}{dx} = 2x \Longrightarrow (x\cos(xy) + 3y^2)\frac{dy}{dx} + y\cos(xy) = 2x$$

   $$\Longrightarrow (x\cos(xy) + 3y^2)\frac{dy}{dx} = 2x - y\cos(xy)$$

   $$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{2x - y\cos(xy)}{x\cos(xy) + 3y^2}}$$

6. $a(t) = t^2 - 7t + 6$ Acceleration

$s(0) = 0, s(1) = 5$ Initial conditions

Find the position function $s(t)$, where $a(t) = s''(t)$

Velocity, $v(t) = \frac{t^3}{3} - \frac{7}{2}t^2 + 6t + C$, C some constant

Position, $s(t) = \frac{t^4}{12} - \frac{7}{6}t^3 + 3t^2 + Ct + D$, D some constant

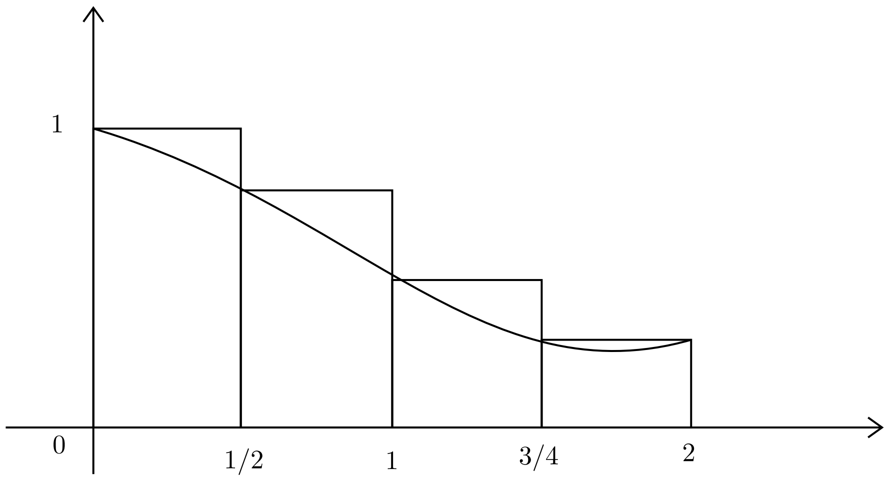
Plug in the initial conditions to find C & D:

$s(0) = 0 + 0 + 0 + 0 + D = 0$ so, $D = 0$

$$s(1) = \frac{1}{12} - \frac{7}{6} + 3 + C = 5 \Longrightarrow C = 5 - 3 - \frac{1}{12} + \frac{7}{6} \Longrightarrow C = \frac{37}{12}$$

So, $\boxed{s(t) = \frac{t^4}{12} - \frac{7}{6}t^3 + 3t^2 + \frac{37}{12}t}$

7.



$$f(x) = \frac{1}{1+x^2},$$

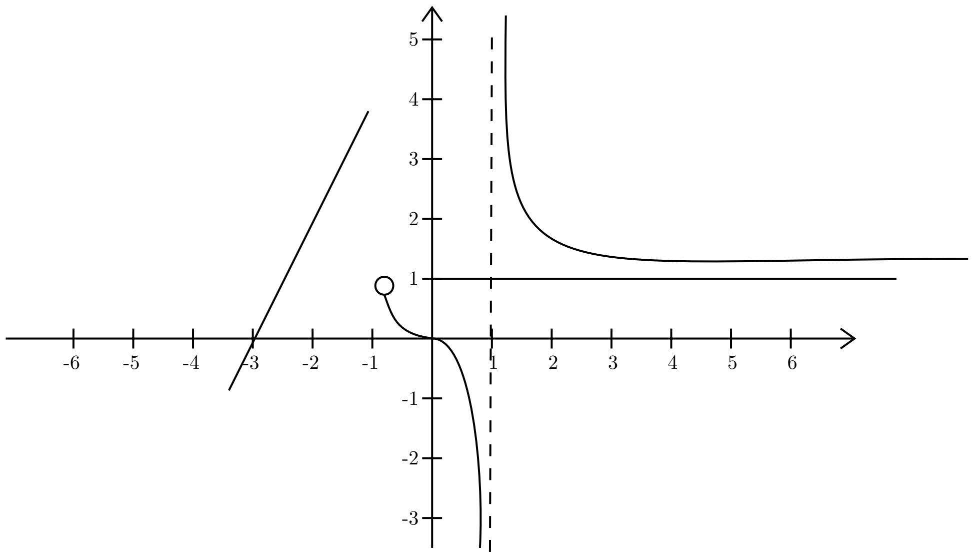
$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$L_4 = (f(x_0) + f(x_1) + f(x_2) + f(x_3))\Delta x = (f(0) + f(\frac{1}{2}) + f(1) + f(\frac{3}{2})).\frac{1}{2}$$

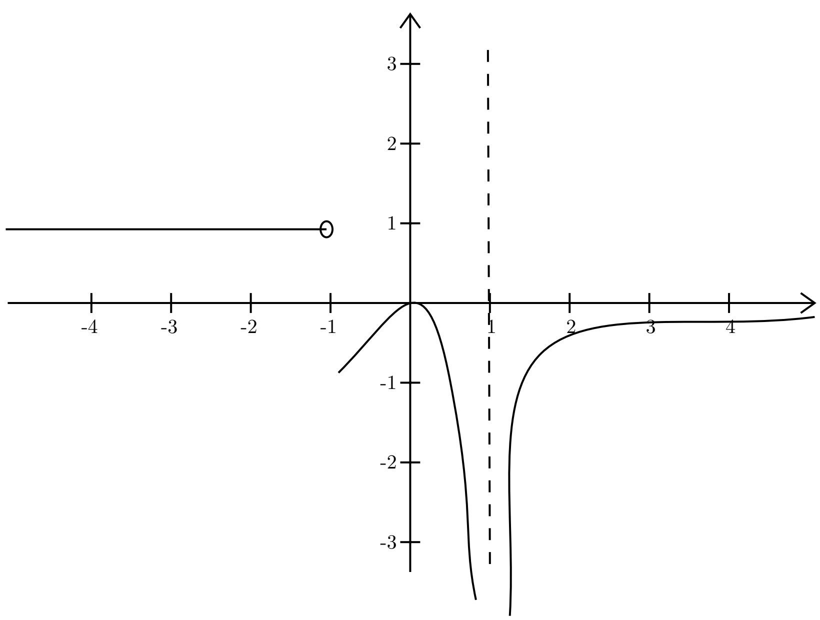
$$\left(\frac{1}{1+0} + \frac{1}{1+\frac{1}{4}} + \frac{1}{1+1} + \frac{1}{1+\frac{3}{2}}\right).(\frac{1}{2}) = (1 + \frac{4}{5} + \frac{1}{2} + \frac{4}{13}).(\frac{1}{2})$$

$$= \boxed{1.304}$$

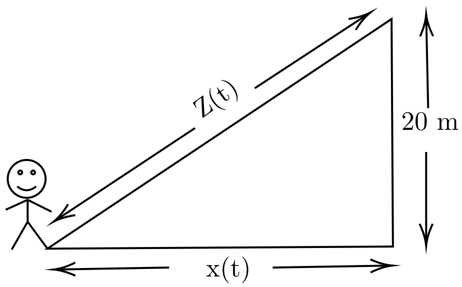
8. $f(x)$:



$f'(x)$:



9.



Let $z(t)$ = distance from the boy's feet to the top of the pole.

$x(t)$ = position of the boy at time t.

$x'(t) = -1$ m/sec.

Find $z'(t)$ When $x(t) = 5$

When $x(t) = 5, \implies z^2 = 5^2 + 20^2 \implies z^2 = 425 \implies z = \sqrt{425}$

We know that $(z(t))^2 = (x(t))^2 + (20)^2$

Differentiate: $2z(t).z'(t) = 2x(t).x'(t)$

$\implies z(t).z'(t) = x(t).x'(t)$

So, $\sqrt{425}.z'(t) = 5.(-1) \implies \boxed{z'(t) = \frac{-5}{\sqrt{425}} m/sec}$.

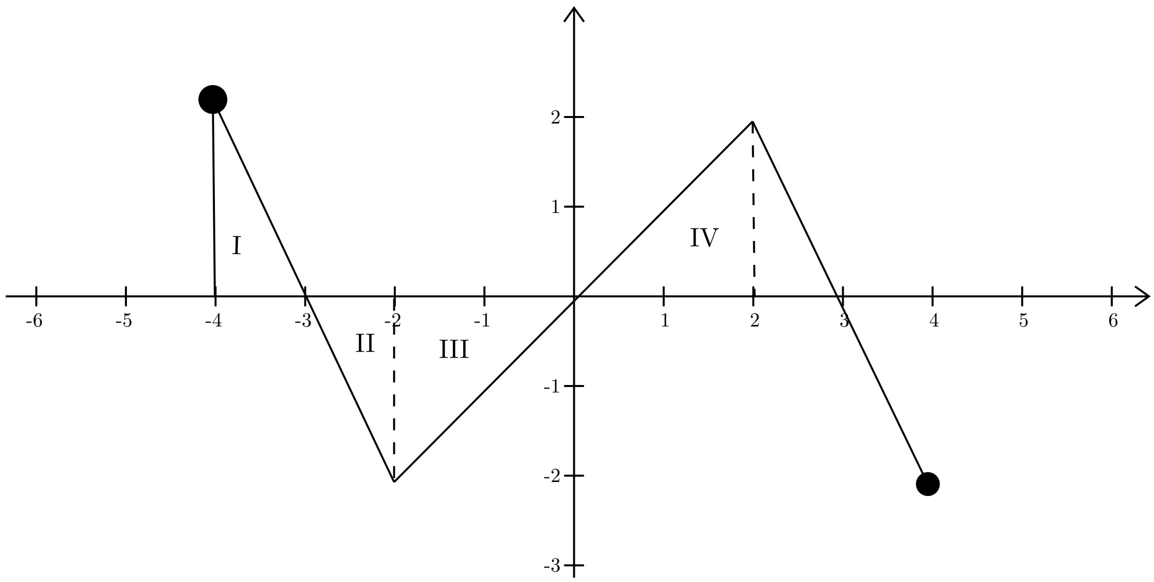
10. (a) $C_{average} = \frac{C(150) - C(100)}{150 - 100} = \frac{(7500 + \sqrt{125(150)}) - (7500 + \sqrt{125(100)})}{50}$

$\frac{\sqrt{125(150)} - \sqrt{125(100)}}{50} \simeq \boxed{0.50 \text{ dollars/bulb}}$.

(b) $C'(x) = \frac{1}{2} \frac{1}{\sqrt{125x}}.125 = \frac{125}{2}.\frac{1}{\sqrt{125x}}$

$C'(125) = \frac{125}{2}.\frac{1}{\sqrt{(125)^2}} = \boxed{\frac{1}{2} \text{dollars/bulb}}$

11.



(a) $g(2) = \int_{-4}^{2} f(t)dt = I + II + III + IV = \boxed{0}$

(Because $I + II = 0 = III + IV$).

(b) $g'(x) = f(x)$ by fundamental theorem of calculus part II.

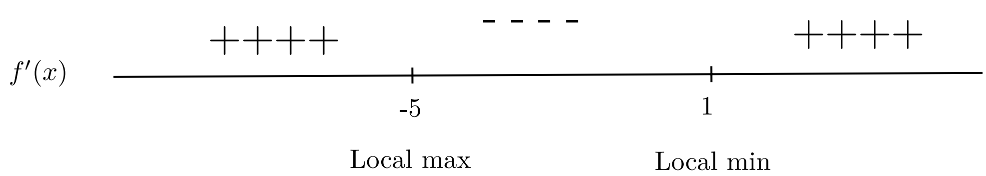
$g'(-2) = f(-2) = \boxed{-2}$.

(c) $g"(x) = f'(x) =$ slope of tangent line to $f(x)$ at x.

So, $g"(0) = f'(0) = \boxed{1}$.

12. Sign chart for $f'(x)$:

$\frac{1}{2}(x+5)(x-1)e^{\frac{x}{2}} = 0 \Longrightarrow x = -5, 1$ critical numbers on $(-\infty, \infty)$

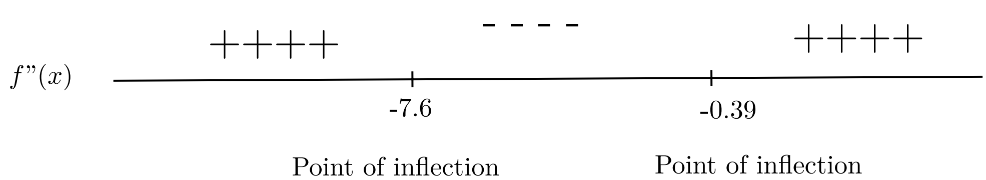


$f(x)$ is increasing on $(-\infty, -5) \cup (1, \infty)$

$f(x)$ is decreasing on $(-5, 1)$

Sign chart for $f''(x)$:

$\frac{1}{4}(x^2+8x+3)e^{\frac{x}{2}} = 0 \Longrightarrow x^2+8x+3 = 0 \Longrightarrow x = \frac{-8-\sqrt{52}}{2}, \frac{-8+\sqrt{52}}{2} \Longrightarrow x \approx -7.6, -0.39$



$f(x)$ is concave down on $(-7.6, 0.39)$

$f(x)$ is concave up on $(-\infty, -7.6) \cup (-0.39, \infty)$

$f(x)$ has a horizontal asymptote at $y = 0$.

Graph: Putting together all of the information solved above and given in the problem: