Problem 1

Set up the difference qoutient:

$$\frac{1}{h}\left(\frac{3}{3(x+h)-2}-\frac{3}{3x-2}\right)=\frac{1}{h}\left(\frac{3}{3x+3h-2}-\frac{3}{3x-2}\right)=\frac{1}{h}\left(\frac{3x-2}{3x-2}\cdot\frac{3}{3x+3h-2}-\frac{3}{3x-2}\cdot\frac{3x+3h-2}{3x+3h-2}\right)$$
$$=\frac{1}{h}\left(\frac{3(3x-2)}{(3x-2)(3x+3h-2)}-\frac{3(3x+3h-2)}{(3x-2)(3x+3h-2)}\right)=\frac{1}{h}\left(\frac{3(3x-2)-3(3x+3h-2)}{(3x-2)(3x+3h-2)}\right)$$
$$=\frac{1}{h}\left(\frac{9x-6-9x-9h+6)}{(3x-2)(3x+3h-2)}\right)=\frac{1}{h}\left(\frac{-9h}{(3x-2)(3x+3h-2)}\right)=\frac{-9}{(3x-2)(3x+3h-2)}$$

So, by definition of the derivative:

$$f'(x)=\lim_{h\to 0}\frac{-9}{(3x-2)(3x+3h-2)}$$

As $h\to 0$ then,

$$\frac{-9}{(3x-2)(3x+3(0)-2)}=\frac{-9}{(3x-2)(3x-2)}=\frac{-9}{(3x-2)^2}$$

Problem 2

Part(a):

$$\lim_{x\to 2}\frac{\frac{1}{x+4}-\frac{1}{3x}}{x-2}=\lim_{x\to 2}\frac{1}{x-2}\left(\frac{3x}{3x}\cdot\frac{1}{x+4}-\frac{1}{3x}\cdot\frac{x+4}{x+4}\right)=\lim_{x\to 2}\frac{1}{x-2}\left(\frac{3x}{3x(x+4)}-\frac{x+4}{3x(x+4)}\right)$$
$$=\lim_{x\to 2}\frac{1}{x-2}\cdot\frac{3x-x-4}{3x(x+4)}=\lim_{x\to 2}\frac{1}{x-2}\cdot\frac{2x-4}{3x(x+4)}=\lim_{x\to 2}\frac{1}{x-2}\cdot\frac{2(x-2)}{3x(x+4)}=\lim_{x\to 2}\frac{2}{3x(x+4)}=\frac{2}{3\cdot 2\cdot 6}=\frac{1}{18}$$

Part(b):

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+3}}{x+7}=\lim_{x\to\infty}\frac{\sqrt{x^2(2+\frac{3}{x^2})}}{x(1+\frac{7}{x})}=\lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{2+\frac{3}{x^2}}}{x(1+\frac{7}{x})}=\lim_{x\to\infty}\frac{|x|\sqrt{2+\frac{3}{x^2}}}{x(1+\frac{7}{x})}$$

Since x is positive as $x\to\infty$, we replace $|x|$ with x.

$$=\lim_{x\to\infty}\frac{x\sqrt{2+\frac{3}{x^2}}}{x(1+\frac{7}{x})}=\lim_{x\to\infty}\frac{\sqrt{2+\frac{3}{x^2}}}{1+\frac{7}{x}}=\frac{\sqrt{2+\frac{3}{"\infty"}}}{1+\frac{7}{"\infty"}}=\frac{\sqrt{2+0}}{1+0}=\sqrt{2}$$

Part(c):

$$\lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2}=\lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2}=\frac{e^4-e^4}{2-2}=\frac{0}{0}$$

This form let's us use L'Hôpital's rule:

$$\lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2}=\lim_{x\to 2}\frac{\left(e^{x^2}-e^4\right)'}{(x-2)'}\lim_{x\to 2}\frac{2xe^{x^2}}{1}=2.2e^{2^2}=4e^4$$

Problem 3

Part(a):

$$f'(x) = \left(\left[\sin(3x^2+x)\right]^4\right)' = 4\left[\sin(3x^2+x)\right]^3\left(\sin(3x^2+x)\right)' = 4\left[\sin(3x^2+x)\right]^3\cos(3x^2+x)\cdot(3x^2+x)'$$

$$= 4\left[\sin(3x^2+x)\right]^3\cos(3x^2+x)\cdot(6x+1)$$

Part(b):

$$g'(x) = (\cos(2x)\ln(x-1))' = \cos(2x)'\ln(x-1)+\cos(2x)\ln(x-1)' = -2\sin(2x)\ln(x-1)+\cos(2x)\frac{1}{\text{x - 1}}$$

Problem 4

Part(a):

$$\int\left(\frac{5}{t^2+1}-\frac{2}{\sqrt{1-t^2}}+\sqrt{2}\right)dt = \int\frac{5}{t^2+1}dt-\int\frac{2}{\sqrt{1-t^2}}dt+\int\sqrt{2}dt = 5\int\frac{1}{t^2+1}dt-2\int\frac{1}{\sqrt{1-t^2}}dt+\int\sqrt{2}dt$$

$$= 5\arctan(t)-2\arcsin(t)+\sqrt{2}t+C$$

Part(b):

$$\int_1^2\left[\frac{1}{x}-\frac{2}{x^3}\right]dx = \int_1^2\frac{1}{x}dx-\int_1^2\frac{2}{x^3}dx = \int_1^2\frac{1}{x}dx-2\int_1^2 x^{-3}dx = \ln|x|\Big|_1^2+x^{-2}\Big|_1^2$$

$$= \ln(2)-\ln(1)+\frac{1}{2^2}-\frac{1}{1^2} = \ln(2)-\frac{3}{4}$$

Problem 5

$$\frac{d}{dx}\left(e^{x-y}\right) = \frac{d}{dx}\left(2x^2-y^2\right) \implies e^{x-y}\cdot\frac{d}{dx}(x-y) = 4x-2y\frac{dy}{dx} \implies e^{x-y}\cdot(1-\frac{dy}{dx}) = 4x-2y\frac{dy}{dx}$$

$$\implies e^{x-y}-e^{x-y}\frac{dy}{dx} = 4x-2y\frac{dy}{dx} \implies 2y\frac{dy}{dx}-e^{x-y}\frac{dy}{dx} = 4x-e^{x-y} \implies \frac{dy}{dx}\left(2y-e^{x-y}\right) = 6x^2-e^{x-y}$$

$$\implies \frac{dy}{dx} = \frac{\text{6x^2} - e^{x-y}}{2y-e^{x-y}}$$

Problem 6

Critical numbers are x-values where $f'(x)=0$ or where $f'(x)$ is undefined. So,

$$f'(x) = \left(\frac{x^2}{x-1}\right)' = \frac{(x^2)'(x-1)-x^2(x-1)'}{(x-1)^2} = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

Then $f'(x)$ is undefined at $x=1$ and $f'(x)=0$ when $x(x-2)=0$ which occurs at $x=0,2$. The critical numbers of $f(x)$ are $x=0,1,2$.

Problem 7

Plug in the known volume and rewrite the volume equation in terms of a single variable h:

$$V = 16\pi = \pi r^2 h \implies 16 = r^2 h \implies \frac{16}{r^2} = h$$

Now we can substitute this in for h in the surface area equation and minimize it by taking the derivative and finding where it's equal to 0:

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{16}{r^2} = 2\pi r^2 + \frac{32\pi}{r}$$

$$S' = 4\pi r - \frac{32\pi}{r^2} = 0 \implies 4\pi r^3 - 32\pi = 0 \implies r^3 = \frac{32\pi}{4\pi} \implies r^3 = 8 \implies r = 2$$

To verify that this occurs at a minimum, make a sign chart or use the second derivative test:

$$S'' = 4\pi + \frac{64\pi}{r^3} \implies S''(2) = 4\pi + \frac{64\pi}{8} > 0$$

The function S is concave up at $r = 2$ so the critical value at $r = 2$ must be a minimum. Therefore, the final dimensions are $r = 2\mathbf{m}$, $h = \frac{16}{2^2} = 4\mathbf{m}$, and the minimum amount of material is $S = 2\pi(2^2) + \frac{32\pi}{2} = 24\pi\mathbf{m}^2$.

Problem 8

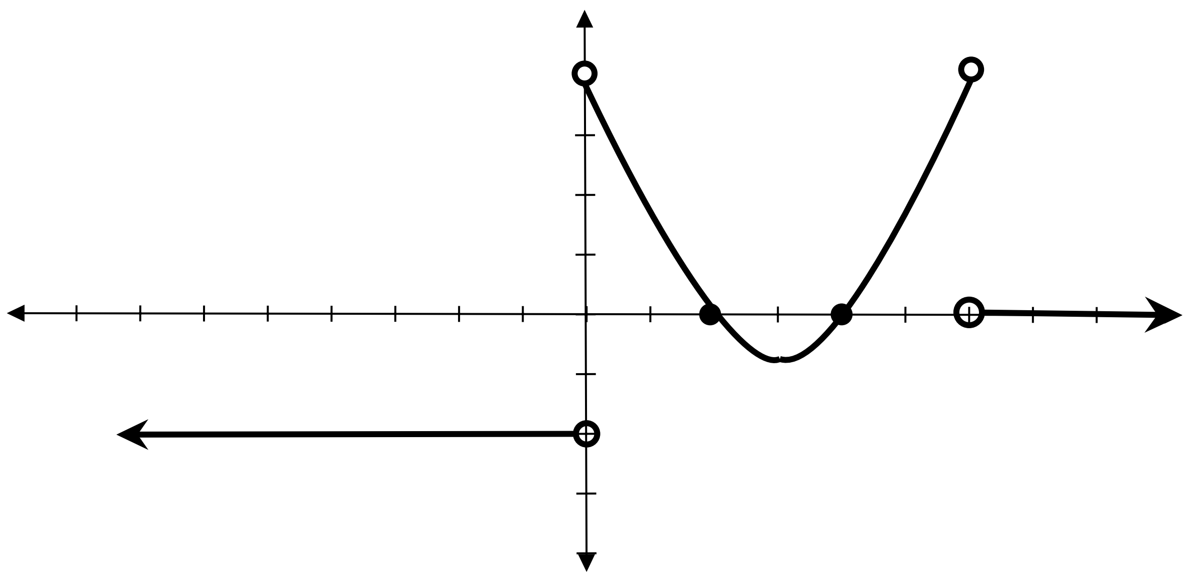
Let T be the total amount of water released between 7a.m. ($t = 0$) and 9:24a.m. ($t = 144$). Then,

$$T = \int_0^{144} r(t)dt = \int_0^{144} \left(100 + \sqrt{t}\right) dt = \left(100t + \frac{2t^{3/2}}{3}\right)\Bigg|_0^{144} = 100(144) + \frac{2\cdot(144)^{3/2}}{3} - \left(100\cdot 0 + \frac{2\cdot 0^{3/2}}{3}\right)$$

$$= 14400 + \frac{2\cdot 12^3}{3} - (0) = 15552 \text{ gallons}$$

Problem 9

The function $f(x)$ has constant slope on $(-\infty, 0)$. Find the slope by using the slope formula $m = \frac{y_2-y_1}{x_2-x_1} = \frac{6-4}{-2-(-1)} = -2$. So, on the graph of $f'(x)$, the function is -2 on the interval $(-\infty, 0)$. Also, the slope of $f(x)$ is zero on the interval $(0, \infty)$. So, the graph of $f'(x)$ will be 0 on the interval $(0, \infty)$. The slope of the graph of $f(x)$ is also zero at $x = 2$ and $x = 4$. So, the graph of $f'(x)$ has points $(2, 0)$ and $(4, 0)$. It is also important to note that $f'(x)$ is not defined at $x = 0$ (because there is a jump) and at $x = 6$ (because it is not smooth).



The graph of $f'(x)$ between the values of $x = 0$ and $x = 6$ are roughly sketched based on the slopes of the original function and do not need to be perfect. Since $f(x)$ looks like a cubic polynomial on the interval $(0, 6)$, a good guess for $f'(x)$ is to draw something that looks quadratic (like a parabola) on $(0, 6)$.

Problem 10

For this problem it is important to recognize the integral of $f(t)$ from -5 to x conceptually is the area under the curve of $f(t)$ between -5 and x. Area above the x-axis is positive and area below the x-axis is negative.

Part(a): $g(5)$ is the are under the curve $f(t)$ from -5 to 5. Break the picture up into triangles and calculate the are of each triangle using $A_\triangle = \frac{1}{2} \cdot \textit{base} \cdot \textit{height}$ and giving the areas a positive or negative sign depending on if they're above or below the x-axis. So if $\triangle_1, \triangle_2, \triangle_3$ are the triangles from left to right:

$$g(5) = -A_{\triangle_1} + A_{\triangle_2} - A_{\triangle_3} = -\frac{1}{2} \cdot 2 \cdot 4 + \frac{1}{2} \cdot 4 \cdot 3 - \frac{1}{2} \cdot 4 \cdot 2 = -4 + 6 - 4 = -2$$

Problem 10 (continued)

Part(b): Using the fundemental theorem of calculus:

$$g'(x) = \frac{d}{dx}\left(\int_{-5}^{x} f(t)dt\right) = f(x)$$

So, $g'(2) = f(2) = -1$ from the graph, and $g''(2) = f'(2) =$ the slope of $f(x)$ at $x = 2$ which is -1.

Part(c): Similar to above, $g'(1) = f(1) = 0$ from the graph, and $g''(1) = f'(1)$ does not exist (DNE) since the derivative is not defined at corners.

Problem 11

Let $f(x) = x^{5/3}$ and $a = 1$. Then $f'(x) = \frac{5}{3}x^{2/3}$ and $f(a) = f(1) = 1$ and $f'(a) = f'(1) = \frac{5}{3}$.

By Linear Approximation:

$$f(x) \approx L(x) = f(a) + f'(a)(x-a) = 1 + \frac{5}{3}(x-1) \implies f(1.2) \approx L(1.2) = 1 + \frac{5}{3}(1.2-1) = \frac{4}{3}$$

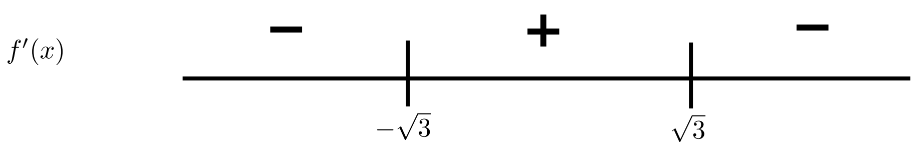
By Differentials:

$$x = 1.2 \implies dx = x - a = 1.2 - 1 = 0.2 \implies dy = f'(a)dx = \frac{5}{3}(0.2) = 1/3$$

$$\text{Therefore, } (1.2)^{3/5} \approx f(a) + dy = 1 + \frac{1}{3} = \frac{4}{3}$$

Problem 12

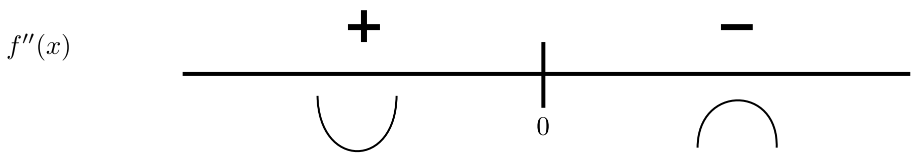
Sign chart for $f'(x)$:
$f'(x) = -\frac{3(x^2-3)}{2(x^2+1)} = 0 \implies -3(x^2-3) = 0 \implies x^2 = 3 \implies x = \pm\sqrt{3}$ (critical numbers)
$f'(x)$ has no "bad" numbers where it is undefined.



So, $f(x)$ is decreasing on the intervals $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.
$f(x)$ is increasing on the interval $(-\sqrt{3}, \sqrt{3})$.
$f(x)$ has a local minimum at $x = -\sqrt{3}$ and a local maximum at $x = \sqrt{3}$.

Problem 12 (continued)

Sign chart for $f''(x)$:
$f''(x) = -\frac{12x}{(x^2+1)^2} = 0 \implies 12x = 0 \implies x = 0$ (critical number)
$f''(x)$ has no "bad" numbers where it is undefined.



So, $f(x)$ is concave down on the interval $(0, \infty)$.
$f(x)$ is concave up on the interval $(-\infty, 0)$.
$f(x)$ has an inflection point at $x = 0$.

Asymptotes: There are no vertical asymptotes. Also, $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$, so there are no horizontal asymptotes.

Graph: Putting together all of the information solved above and given in the problem: