MAT 2010 Fall'22

1. Apply the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{5x+5h-4}-\sqrt{5x-4}}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{5x+5h-4}-\sqrt{5x-4}}{h} \cdot \frac{\sqrt{5x+5h-4}+\sqrt{5x-4}}{\sqrt{5x+5h-4}+\sqrt{5x-4}} =$$

$$\lim_{h\to 0} \frac{(5x+5h-4)-(5x-4)}{h[\sqrt{5x+5h-4}+\sqrt{5x-4}]} =$$

$$\lim_{h\to 0} \frac{5h}{h[\sqrt{5x+5h-4}+\sqrt{5x-4}]} =$$

$$\lim_{h\to 0} \frac{5}{\sqrt{5x+5h-4}+\sqrt{5x-4}} =$$

$$\frac{5}{\sqrt{5x-4}+\sqrt{5x-4}} = \boxed{\frac{5}{2\sqrt{5x-4}}}.$$

2. (a) $$\lim_{x\to 2} \frac{\frac{1}{x+4}-\frac{1}{3x}}{x-2} = \lim_{x\to 2} \frac{\frac{3x-(x+4)}{3x(x+4)}}{x-2} = \lim_{x\to 2} \frac{3x-x-4}{3x(x+4)(x-2)} = \lim_{x\to 2} \frac{2(x-2)}{3x(x+4)(x-2)} =$$

$$\lim_{x\to 2} \frac{2}{3x(x+4)} = \frac{2}{3(2)(2+4)} = \boxed{\frac{1}{18}}.$$

(b) $$|x-5| = \begin{cases} \text{x}-5 & \text{if } x \geq 5 \\ -(\text{x}-5) & \text{if } x < 5 \end{cases}$$

$$\lim_{x\to 5^-} \frac{|x-5|}{(x-5)^2} = \lim_{x\to 5^-} \frac{-(x-5)}{(x-5)^2} = \lim_{x\to 5^-} \frac{-1}{x-5} = \boxed{\infty}.$$

(c) $\lim_{x\to-\infty} \frac{x^2}{e^{1-x}} \left[\frac{\infty}{\infty} form\right]$, by l'Hopital rule.

$= \lim_{x\to-\infty} \frac{2x}{-e^{1-x}} \left[\frac{\infty}{\infty} form\right]$, by l'Hopital rule.

$= \lim_{x\to-\infty} \frac{2}{e^{1-x}} = \boxed{0}$.

3. (a) $f'(x) = \left(\sqrt[3]{x^2}.\left(x+3\right)^5\right)' = \boxed{\frac{2}{3}x^{-\frac{1}{3}}\left(x+3\right)^5 + 5\left(x+3\right)^4.x^{\frac{2}{3}}}$.

   (b) $h'(x) = \left(\ln\left(\arctan(3x)\right)\right)' = \frac{1}{\arctan 3x}.\frac{1}{1+(3x)^2}.3 = \boxed{\frac{3}{(1+9x^2)\arctan(3x)}}$.

4. (a) $\int \left(\sqrt{2}\sec(x)\tan(x) + \sec^2(x) - \frac{6}{\sqrt{1-x^2}}\right)dx = \boxed{\sqrt{2}\sec x + \tan x - 6\arcsin x + C}$.

   (b) $\int_1^e \left(x - \frac{1}{x}\right) = \frac{x^2}{2} - \ln|x|\Big]_1^e = \frac{e^2}{2} - \ln|e| - \left(\frac{1}{2} - \ln(1)\right) = \boxed{\frac{e^2}{2} - \frac{3}{2}}$.

5. (a) $\lim_{x\to 2} f(x) = \boxed{DNE}$.

   (b) $\lim_{x\to\infty} f(x) = \boxed{-1}$.

   (c) $\lim_{x\to 3^-} f(x) = \boxed{0}$.

   (d) $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \boxed{\frac{3}{2}}$.

   (e) $\lim_{x\to 0} f(x) = \boxed{-1}$.

6. $x^2 + 2xy + 4y^2 = 13$

(a) Differentiate both sides with respect of x.

$$2x + 2y + 2xy' + 8yy' = 0$$

$$2xy' + 8yy' = -2x - 2y$$

$$2(x + 4y)y' = -2(x + y)$$

$$y' = \frac{-2(x + y)}{2(x + 4y)}$$

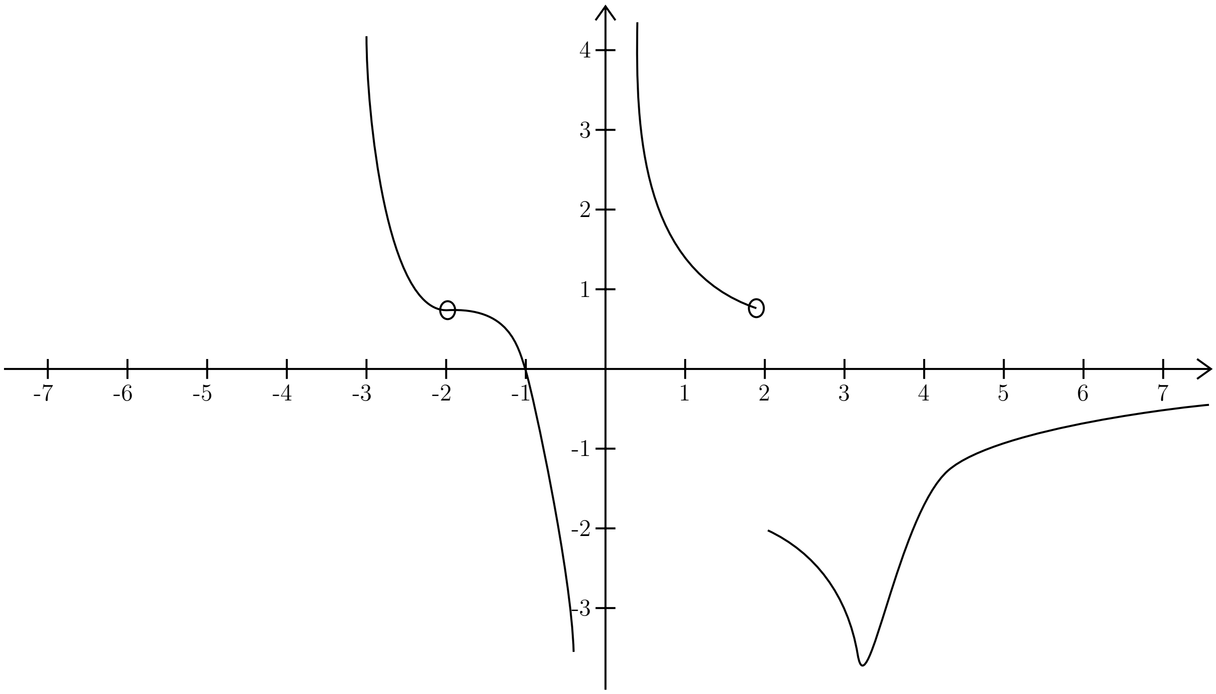
$$\boxed{y' = -\frac{(x + y)}{(x + 4y)}}$$

(b) Equation of tangent at (-1,2).

$$m = -\frac{-1 + 2}{-1 + 8} = -\frac{1}{7}$$

$$y - y_1 = m(x - x_1) \longrightarrow y - 2 = -\frac{1}{7}(x + 1) \longrightarrow \boxed{y = -\frac{1}{7}x + \frac{13}{7}}$$

7.



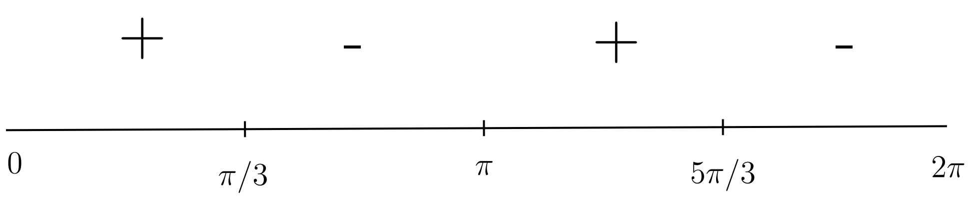
8. $s(t) = \sin^2 t + \cos t$

$v(t) = s'(t) = 2 \sin t \cos t - \sin t$

$v(t) = 0 \Longrightarrow \sin t(2 \cos t - 1) = 0$

$\Longrightarrow \sin t = 0 \longrightarrow \boxed{t = 0, \pi, \frac{\pi}{2}}$

OR $\Longrightarrow 2 \cos t - 1 = 0 \longrightarrow \cos t = \frac{1}{2} \longrightarrow \boxed{t = \frac{\pi}{3}, \frac{5\pi}{3}}$.



Particle is moving forward on $(0, \frac{\pi}{3}) \cup (\pi, \frac{5\pi}{3})$,

OR particle is moving forward when $0 < t < \frac{\pi}{3}$ or $\pi < t < \frac{5\pi}{3}$.

9. $f''(t) = 3\sqrt{t} + 2, \qquad f(0) = 7, \qquad f'(0) = 3.$

$$f'(t) = \int (3\sqrt{t} + 2)dt = 3.\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 2t + C = 2t^{\frac{3}{2}} + 2t + C$$

$f'(0) = 2(0) + 2(0) + C = 3 \Longrightarrow C = 3$ So, $\boxed{f'(t) = 2t^{\frac{3}{2}} + 2t + 3}$

$$f(t) = \int (2t^{\frac{3}{2}} + 2t + 3)dt = 2.\frac{t^{\frac{5}{2}}}{\frac{5}{2}} + 2.\frac{t^2}{2} + 3t + D = \frac{4}{5}t^{\frac{5}{2}} + t^2 + 3t + D$$

$$f(0) = \frac{4}{5}(0) + 0 + 3(0) + D = 7 \Longrightarrow D = 7$$

$\boxed{f(t) = \frac{4}{5}t^{\frac{5}{2}} + t^2 + 3t + 7}$.

10. Let $f(x) = \sqrt{x}, \qquad a = 16, \qquad x = 14.9.$

- Using Differential:

$dx = x - a = 14.9 - 16 = -1.1$

$f'(x) = \frac{1}{2\sqrt{x}} \qquad f(16) = 4 \;\; \& \;\; f'(16) = \frac{1}{8}$

$$dy = f'(x)dx = \frac{1}{2\sqrt{x}}dx$$

At $a = 16$ $\qquad dy = \frac{1}{8}.(-1.1) = -0.1375$

Now at $\sqrt{14.9} \approx f(a) + dy = 4 + (-0.1375) = \boxed{3.86}$.

- Using linear approximation:

$$dx = x - a = 14.9 - 16 = -1.1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad f(16) = 4 \ \ \& \ \ f'(16) = \frac{1}{8}$$

$$L(x) = f(a) + f'(a)(x - a) = 4 + \frac{1}{8}(x - 16)$$

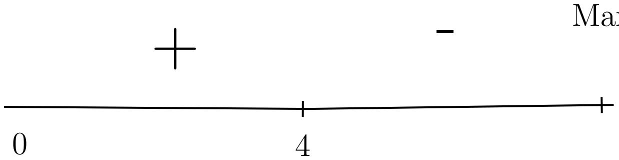
$$\sqrt{14.9} \approx L(14.9) = 4 + \frac{1}{8}(14.9 - 16) = 4 + \frac{1}{8}(-1.1) = \boxed{3.86}.$$

11. $C(t) = \dfrac{2t}{16 + t^2} \qquad t > 0.$

$$C'(t) = \frac{2.(16 + t^2) - (2t).(2t)}{(16 + t^2)^2} = \frac{32 + 2t^2 - 4t^2}{(16 + t^2)^2} = \frac{32 - 2t^2}{(16 + t^2)^2}$$

$$C'(t) = 0 \Longrightarrow 32 - 2t^2 = 0 \Longrightarrow t = 4(t > 0)$$

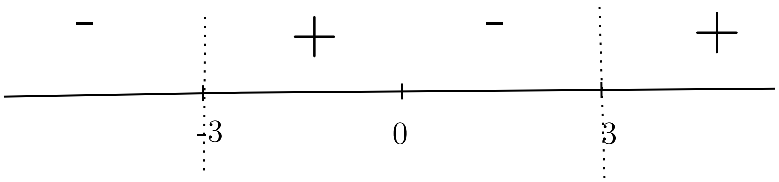
Max Value at t = 4



The concentration will have a maximum value after $\boxed{4}$ hours.

12. $f'(x) = \dfrac{2x}{x^2-9}$

$2x = 0 \Longrightarrow x = 0 \;\;\&\;\; x^2 - 9 = 0 \Longrightarrow x = \pm 3.$

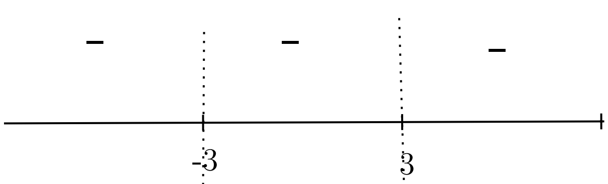


f is increasing on $\boxed{(-3,0)\cup(3,\infty)}$,f is decreasing on $\boxed{(-\infty,-3)\cup(0,3)}$.

Local maximum at $\boxed{x=0}$, No local minimum.

$f''(x) = -\dfrac{2(x^2+9)}{(x^2-9)^2}$

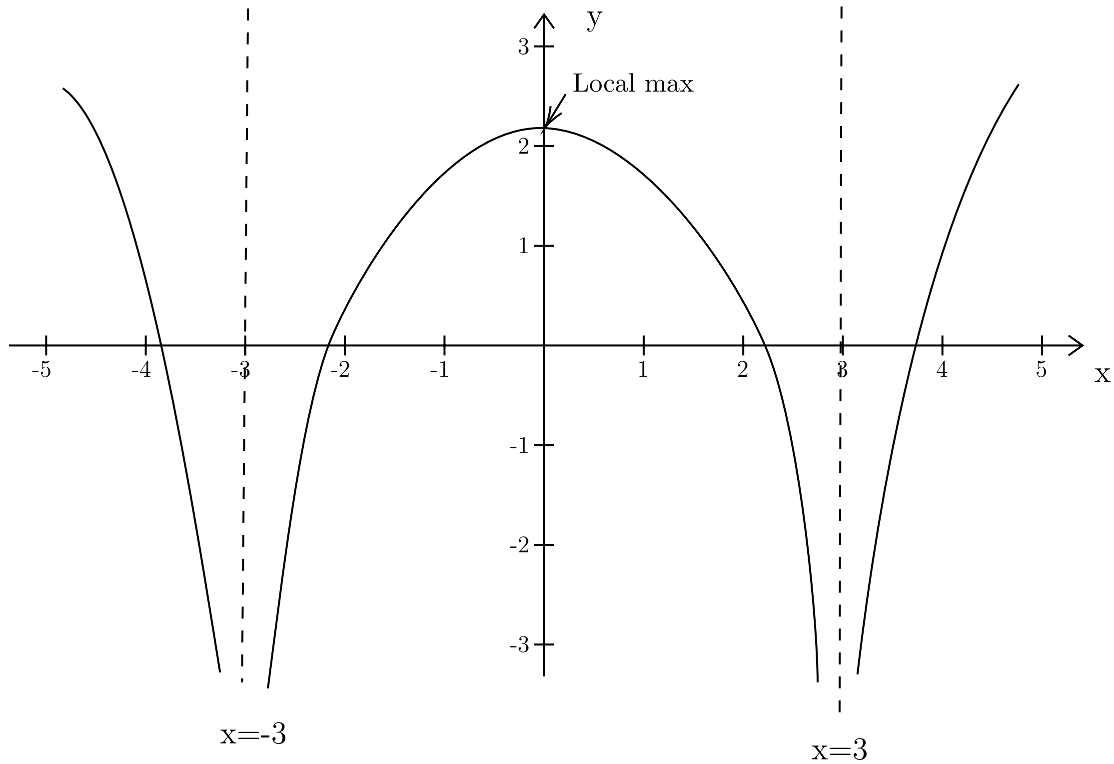
$x^2 + 9 = 0$ No solution $\;\&\;$ $x^2 - 9 = 0 \Longrightarrow x = \pm 3$



f is concave down on $\boxed{(-\infty,-3)\cup(-3,3)\cup(3,\infty)}$, f is not concave down.

No inflection points.

Vertical asymptotes: $x = 3, x = -3$, No Horizontal asymptotes.


y
Local max
x
3
2
1
-1
-2
-3
-5
-4
-3
-2
-1
1
2
3
4
5
x=-3
x=3