Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

WEDNESDAY, JANUARY 4, 2006
9:00 A.M. — 1:00 P.M.

ROOM 245 PHYSICS BUILDING

INSTRUCTIONS: This paper contains six problems. The first four problems are worth 10 points per problem and the last two are worth 15 points. You are to solve all six using a separate booklet for each problem. On the front cover of each booklet, you must write the following information.

1) Your special ID number that you obtained from Delores Cowen
2) The problem number and the title of the exam (i.e. problem #1, part #1)
3) Please press hard and make your answers legible.

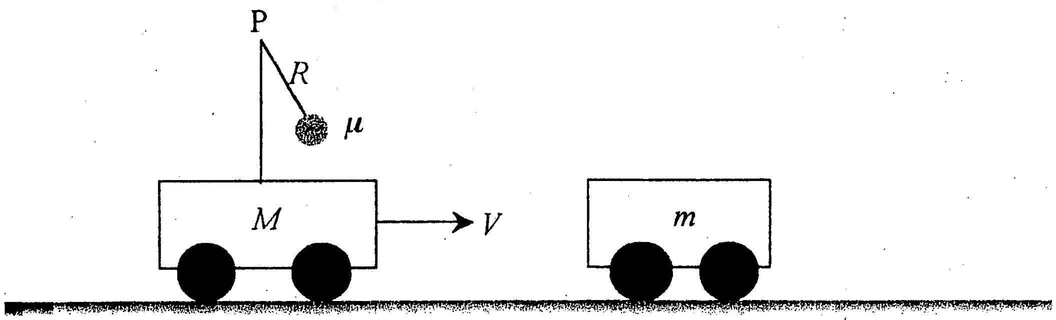
Please do NOT write your name on the cover or anywhere else in the booklet!

1. A cart of mass M has a pole mounted on it as illustrated in the figure below. Assume the pole mass is negligible. A ball of mass μ hangs by a massless string, of length R, attached to the pole at point P.

a. Suppose that the cart (of mass M) and the ball are initially at rest, with the ball hanging in its equilibrium position. Calculate the minimum velocity that must be imparted to the ball for it to rotate in a circle of radius R in the vertical plane. (2 pts.)

b. Now suppose the cart and ball have initial velocity V. The cart crashes into another cart of mass m and sticks to it. Find the velocity of the system after a collision. In this part and the next, neglect friction and assume that $M, m >> \mu$. (2 pts.)

c. Find the smallest value of the initial cart velocity for which the ball can go in circles in the vertical plane following a collision. (6 pts.)

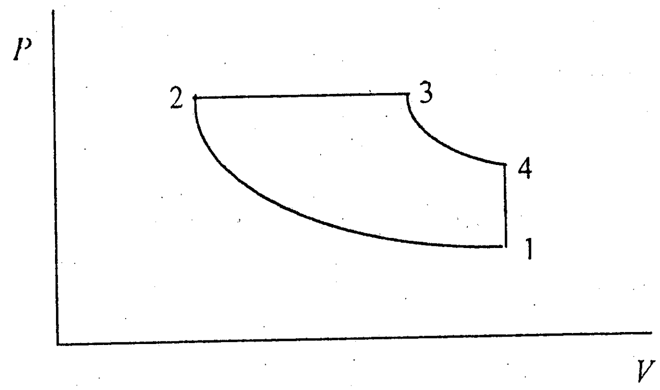


2. Consider the Diesel cycle shown below. Assume that the paths $1 \rightarrow 2$ and $3 \rightarrow 4$ are adiabats. Also assume the heat capacities c_p and c_v are constant and temperatures and volumes at points 1, 2, 3, and 4 are known.

a. Calculate the work done in a cycle. (4 pts.)

b. Determine the total heat supplied in a cycle. (3 pts.)

c. Obtain the efficiency in terms of the volumes V_1, V_2, V_3. (3 pts.)



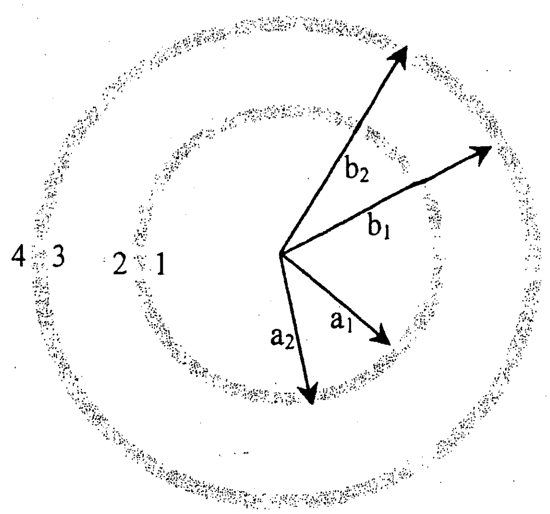
3. Two concentric conducting spherical shells are arranged as shown in the figure below. Each shell has a nonzero thickness. The inner shell has the inner and outer radii $a_1 < a_2$ and the outer shell has $b_1 < b_2$.

a. A charge Q_1 is placed on the inner shell, while a charge Q_2 is placed on the outer shell. Find the charge density on each of the four surfaces labeled in the figure. (3 pts.)

b. If $Q_2 = -Q_1$, find the mutual capacitance of the system. (2 pts.)

c. Suppose now that the space between the spheres is filled with an insulating material of dielectric constant ε. What are the surface charge densities and polarization charge densities for arbitrary Q_1 and Q_2. (3 pts.)

d. As in part b, take $Q_2 = -Q_1$ and find the mutual capacitance of the system with the dielectric. (2 pts.)



4. Two spin ½ particles are separated by a fixed distance $\vec{a} = a\hat{z}$ and interact only through the dipole interaction

$$H = \frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{a^3} - 3\frac{(\vec{\mu}_1 \cdot \vec{a})(\vec{\mu}_2 \cdot \vec{a})}{a^5},$$

where $\vec{\mu}_i$ are the magnetic moments of particles $i = 1$ and 2 due to their spins. The system of two spins consists of eigenstates of S^2 and total S_z, where $\vec{S}$ is the total spin.

a. Write the Hamiltonian in terms of the spin operators for each particle. (3 pts.)

b. Write the Hamiltonian in terms of spin S^2 and S_z. (4 pts.)

c. Give the eigenvalues for all states. (3 pts.)

5. (15 pts.) Consider a one-dimensional semi-infinite square well potential for E < 0

$$V(x) = \begin{cases} \infty; & x < 0 \\ -V_0; & 0 < x < a \\ 0; & x > a \end{cases}$$

a. Write down the time independent Schrödinger equation for this potential. Obtain the boundary conditions for the wave function at $x = 0$ and $x = a$. (2 pts.)

b. Derive the spatial wave function for a particle in this potential. You need not determine the overall normalization constant. (5 pts.)

c. Obtain the equation that determines the allowed values of E. (4 pts.)

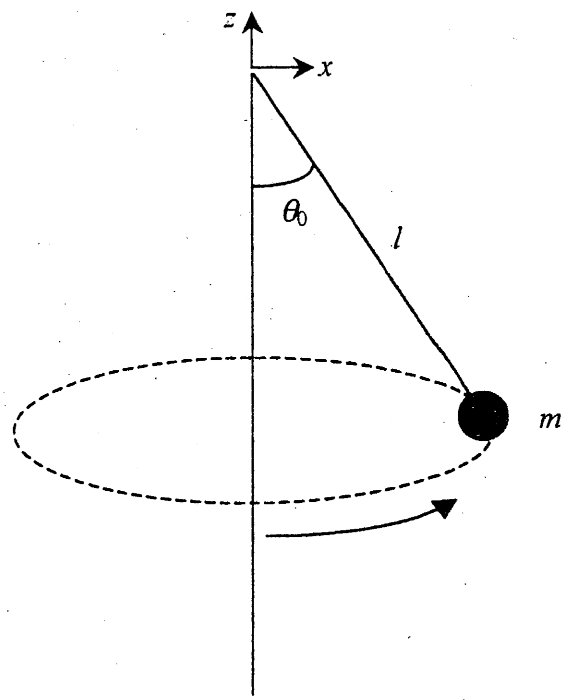
d. Estimate the energy eigenvalues when the potential is wide and shallow, i.e., $V_0 << \hbar^2/2ma^2$. (4 pts.)

6. (15 pts.) A spherical pendulum consists of a point mass m tied by a massless string of length l to a fixed point, so that it is constrained to move on a spherical surface as shown below.

a. Derive the Lagrangian for the system. Find the conserved quantities and the equations of motion. (5 pts.)

b. At what angular velocity will it move on a circle such that the string makes a constant angle θ_0 with the vertical? (5 pts.)

c. Suppose that the mass is initially in the circular orbit described in part b. At $t = 0$ it receives an impulse perpendicular to its velocity, resulting in an orbit which has its highest point such that the string makes an angle $\theta_1 > \theta_0$ with the vertical. Obtain – but do not solve – the equation that can be used to determine the angle θ_2 of the string when the mass is at its lowest point. (5 pts.)



Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART II

FRIDAY, JANUARY 6, 2006
9:00 A.M. — 1:00 P.M.

ROOM 245 PHYSICS BUILDING

INSTRUCTIONS: This paper contains six problems. The first four problems are worth 10 points per problem and the last two are worth 15 points. You are to solve all six using a separate booklet for each problem. On the front cover of each booklet, you must write the following information.

1) Your special ID number that you obtained from Delores Cowen
2) The problem number and the title of the exam (i.e. problem #1, part #II)
3) Please press hard and make your answers legible.

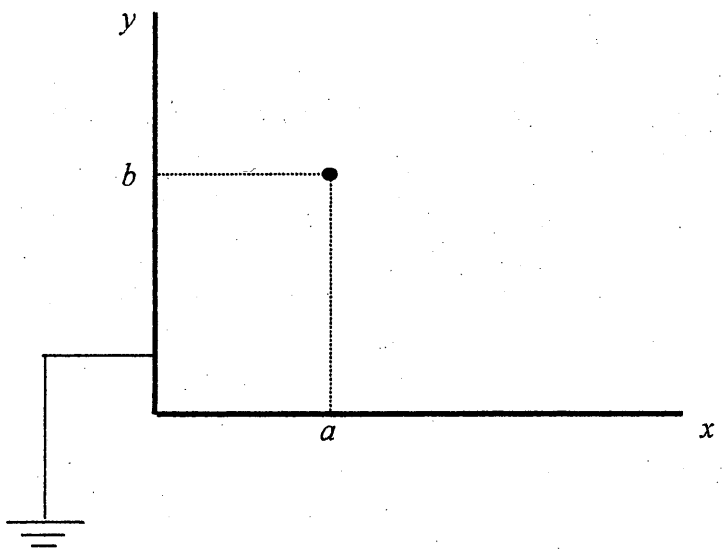
Please do NOT write your name on the cover or anywhere else in the booklet!

1. Two semi-infinite grounded conducting planes meet at right angles. A point charge q is situated in the region between them.

a. Find the potential between the planes using the method of images. (4 pts.)

b. What is the force on q? (3 pts.)

c. How much work does it take to bring q in from infinity? (3 pts.)



2. Consider the thermodynamic properties of an ensemble of diatomic molecules of moment of inertia I that are free to rotate in two dimensions. Each individual molecule has quantized energy levels given by

$$\varepsilon = \frac{\hbar^2}{2I} J^2, \qquad J = 0, 1, 2, \ldots,$$

with degeneracies $g_J = 2$ for $J > 0$ and $g_J = 1$ for $J = 0$.

a. Assuming that the temperature is high enough that $k_B T >> \hbar^2 / 2I$, derive the partition function Z_{rot} for an individual diatomic molecule in two dimensions. (6 pts.)

b. Determine the thermodynamic energy U in this high-temperature limit for a set of N indistinguishable, independent, diatomic molecules. (2 pts.)

c. Find the heat capacity C_v for this system. (2 pts.)

3. Consider the electromagnetic field

$$\vec{E}(x,y,t) = E_0 \cos(\pi x/L)\cos(\pi y/L)\sin(\omega t)\,\hat{z}$$

$$\vec{B}(x,y,t) = B_0\left[-\cos(\pi x/L)\sin(\pi y/L)\,\hat{x} + \sin(\pi x/L)\cos(\pi y/L)\,\hat{y}\right]\cos(\omega t)$$

a. Show that this field satisfies the Maxwell equations in vacuum if $\omega = \sqrt{2}\pi c/L$ and $B_0 = E_0/\left(\sqrt{2}c\right)$. (6 pts.)

a. This field represents a standing electromagnetic wave inside a box with metal walls and a square cross section of size $L \times L$ parallel to the xy plane, and very long in the z direction. This is and example of a cavity oscillator. Sketch the E and B fields. Note that the wave number satisfies $k = \pi/L$, so that the wavelength is $2L$. (4 pts.)

4. Consider a simple harmonic oscillator in one dimension. Introduce raising and lowering operators, a and a^+, so that the Hamiltonian is

$$H = \hbar\omega\left(a^+a + \frac{1}{2}\right).$$

Suppose that the initial state is

$$\Psi(0) = \frac{1}{\sqrt{5}}\psi_1 + \frac{2}{\sqrt{5}}\psi_2,$$

where ψ_n are the energy eigenfunctions of energy $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ and $a\psi_n = \sqrt{n}\,\psi_{n-1}$.

a. What is the wave function $\Psi(t)$ in terms of these eigenfunctions? (4 pts.)

b. What is the expectation value for the energy? (3 pts.)

c. The position x operator can be represented by $x = X_0(a + a^+)$ where $X_0 = \sqrt{\hbar/2m\omega}$. Derive an expression for the time dependent expectation value of position $\langle x(t)\rangle$. (3 pts.)

5. (15 pts.) A quadrupole magnet can be constructed from two circular wire loops placed as shown in the figure below. The loops carry equal currents I flowing in opposite directions. (Note that this situation differs from a Helmoltz coil, in which the currents flow in the same direction).

a. First consider a *single* circular loop of wire of radius R carrying current I. Calculate the magnetic field along the symmetry z-axis. (4 pts.)

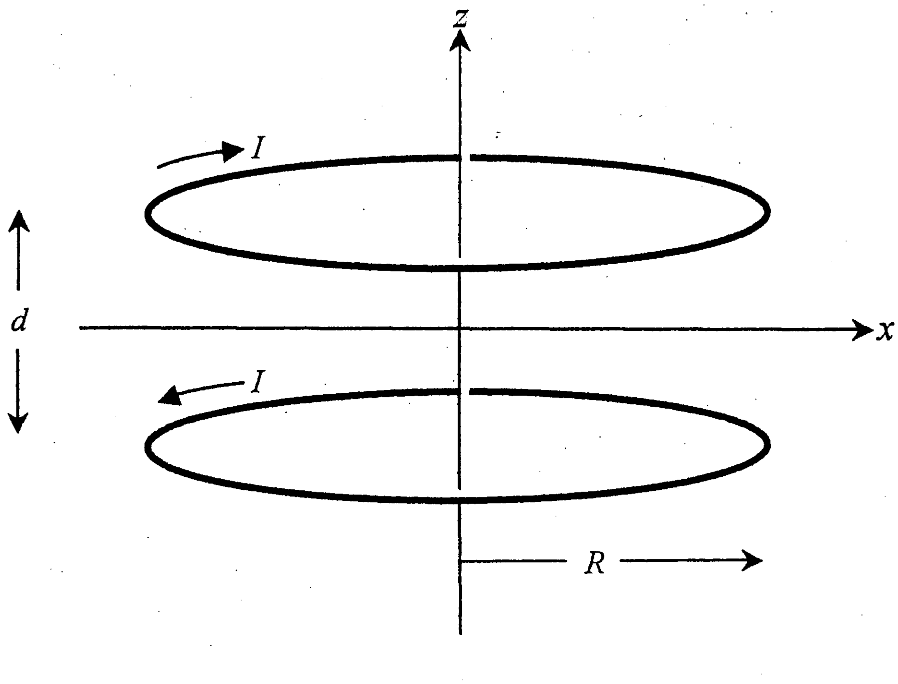
b. Now consider two loops at the respective positions $z = \pm d/2$ as in the figure. Show that the net $B_z \propto z$ for $z << R$ and find the coefficient. (4 pts.)

c. Sketch the magnetic field in the zx and xy planes. (3 pts.)

d. Use $\vec{\nabla} \cdot \vec{B} = 0$ and symmetry arguments to show that the magnitude of the field satisfies

$$B = \alpha\left(x^2 + y^2 + 4z^2\right)^{1/2}$$

for points (x, y, z) near the origin, i.e., $x, y, z << d$ and R. Find the coefficient α. (4 pts.)



6. Consider an electron in a uniform magnetic field along the z direction. A measurement shows that the spin is along the positive y direction at $t = 0$.

a. Find the eigenvector describing the initial spin state, taking z as the quantization axis. (5 pts.)

b. Obtain the Hamiltonian and the differential equation describing the time evolution of the spin state (i.e., the spinor wave function) describing the electron in the magnetic field. (5 pts.)

c. Find the state vector at time $t > 0$ and compute the expectation value of s_x at time t. (5 pts.)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$