Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

MONDAY, JANUARY 8, 2007
9:00 A.M. — 1:00 P.M.

ROOM 245 PHYSICS BUILDING

INSTRUCTIONS: This paper contains six problems. The first four problems are worth 10 points per problem and the last two are worth 15 points. You are to solve all six using a separate booklet for each problem. On the front cover of each booklet, you must write the following information.

1) Your special ID number that you obtained from Delores Cowen
2) The problem number and the title of the exam (i.e. problem #1, part #1)
3) Please press hard and make your answers legible.

Please do NOT write your name on the cover or anywhere else in the booklet!

1. A rocket is projected straight up and explodes into three equally massive fragments just as it reaches the top of its flight. One of the fragments is observed to come straight down in a time t_1, while the other two land at a time t_2 after the burst. Neglect air resistance.

a. (6 pts) Find the height $h(t_1,t_2)$ at which the fragmentation occurred.
b. (4 pts) Find the horizontal distance $L(t_1,t_2)$ (from the point of fragmentation) at which the other two fragments fall to the ground.

2. A thin circular disk of radius R in vacuum has a fixed total charge Q. The disk lies in the x-y plane.

a. (4 pts) Find the electrostatic potential everywhere on the positive z-axis.

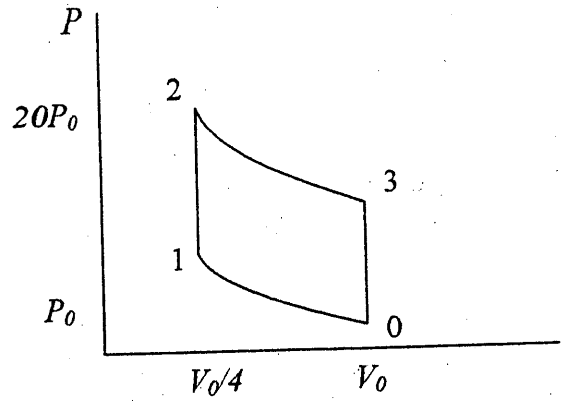
b. (6 pts) In this problem, the electrostatic potential everywhere in space exhibits azimuthal symmetry. Thus, we may write $\Phi = \Phi(r,\theta)$, where r is the radial distance from the origin, and θ is the polar angle. Φ satisfies Laplace's equation $\nabla^2\Phi = 0$ and in these coordinates, the general solution with azimuthal symmetry has the form:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta).$$

Use your solution found in part (a) and the form of the general solution above to find a unique expression for $\Phi = \Phi(r,\theta)$ valid in the half space $z > 0$ or, equivalently, in the polar angle range $0 \le \theta < \pi/2$.

Hint: $P_l(1) = 1$

3. (10pts) An ideal gas engine consists of two adiabats and two isochores (see Figure below). Starting at point 0 where $P = P_o$ and $V = V_o$ it is compressed adiabatically to one fourth of its original volume (point 1), then heated at fixed volume until it reaches twenty times its original pressure (point 2), next expanded adiabatically to its original volume (point 3), and finally cooled at fixed volume until it reaches its original pressure (point 0). Assuming that the ideal gas is air calculate the efficiency of this engine and compare its efficiency to the efficiency of a Carnot engine operating between the same highest and lowest temperatures. (The engine is the so-called Otto cycle, an approximate model of the internal combustion engine).



4. Two identical spin one-half fundamental particles of mass m are in a one-dimensional infinite square well: $V(x) = 0$, for $0 < x < L$, and $V(x) = \infty$ otherwise. The total spin of the two particles is $S = 0$.

a. (2 pts) Are these particles fermions, bosons or neither?

b. (4 pts) What is the energy of the ground state of the system?

c. (4 pts) Write the complete wavefunction for this two-particle ground state.

5.

a. (3pts) Derive the commutator $[p_x, x]$ (where p_x is the x-component of the momentum operator in the x-representation)

b. (3pts) Consider a particle of mass m in a one-dimensional harmonic oscillator potential with the classical angular frequency ω. Write the Hamiltonian for this system, using the operator p_x and x.

c. (3pts) Show that the Hamiltonian found in part b can be written in the form $H = (a_- a_+ - \frac{1}{2}\hbar\omega)$, where $a_\pm = \alpha p_x \pm \beta x$, by finding α and β.

d. (3pts) Find the commutator [a₋, a₊] using your answers to (a) and (c).

e. (3pts) If ψ_n are the eigenstates of the Hamiltonian corresponding to eigenvalue E_n, i.e., $H\psi_n = E_n\psi_n$, show that a± are raising and lowering operators, respectively. That is, show that $a_+\psi_n = c_+\psi_{n+1}$ and $a_-\psi_n = c_-\psi_{n-1}$ where c_+ and c_- are constants that you do not need to determine. Hint: What is E_n?

6. A long uniform non-conducting cylinder of radius R carries a charge per unit length λ distributed uniformly throughout its volume. The cylinder is wound with N turns per unit length of wire carrying current I. This current produces a magnetic field which we assume to be uniform throughout the cylinder but zero outside.

a. (3 pts) Determine both the electric and magnetic fields within the cylinder.

b. (3 pts) Determine the Poynting vector $\vec{S}$ as a function of distance r from the axis of the cylinder and specify its direction.

c. (3 pts) The linear momentum density of an electromagnetic field is $\vec{S}/c^2$. Find the angular momentum per unit length of the electromagnetic field about the axis of the cylinder.

d. (3 pts) The current is now turned off at a constant rate $\frac{dI}{dt} = -\alpha$ (constant). Find the torque per unit length exerted on the cylinder.

e. (3 pts) Assuming the cylinder is free to rotate about its axis, compute the total angular momentum of the cylinder per unit length after the current has been reduced to zero. Compare your result to (c).

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PART II

WEDNESDAY, JANUARY 10, 2007
9:00 A.M. — 1:00 P.M.

ROOM 245 PHYSICS BUILDING

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1. Consider a solid cylinder of mass m and radius r sliding without rolling down a smooth inclined face of mass M that is free to move on a horizontal plane without friction. The moment of inertia of a cylinder is $I = \frac{1}{2}mr^2$.

   a. (3 pts) How far has the wedge moved by the time the cylinder has descended from rest a vertical distance of h?

   b. (3 pts) Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case?

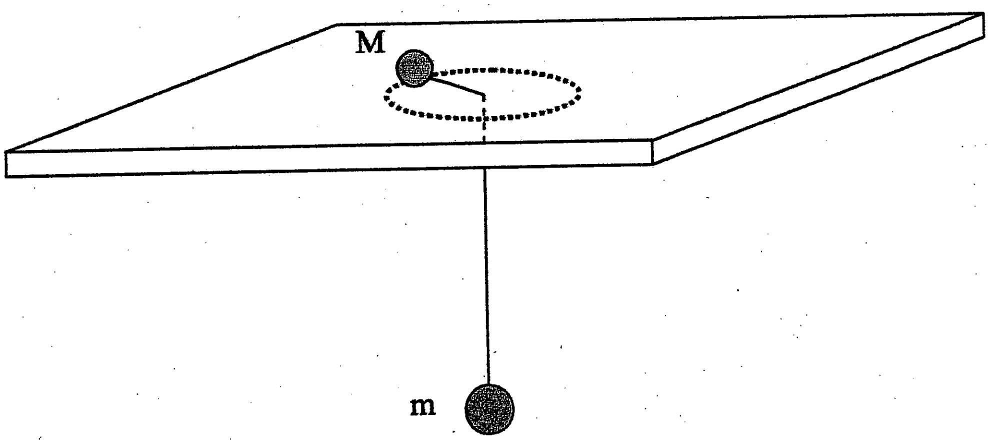
   c. (4 pts) In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

2. Two point masses M and m are connected by a string of length l passing through a hole in a horizontal table of negligible thickness. The particle of mass M is constrained to move on a horizontal plane, while the particle of mass m is constrained to a vertical line (see Figure). The motion is frictionless.

a. (2 pts) What initial velocity must mass M be given so that mass m will remain motionless a distance d below the surface of the table?

b. (4 pts) Find the Lagrangian of the system and derive the equation of motion. Obtain the result of part a) from this equation.

c. (4 pts) If m is slightly displaced in a vertical direction, small oscillations ensue. Use Lagrange's equations to find the period of these oscillations. Are these oscillations stable?



3. An ideal gas consists of N molecules per unit volume; each molecule with a constant electric dipole $\boldsymbol{p}$. Neglect any interaction between the molecules.

a. (7 pts) Calculate the electric polarization $\boldsymbol{P}$ of this ideal gas in a uniform external electric field $\boldsymbol{E}$ at temperature T.

b. (3 pts) Find the dielectric constant of this gas in small fields.

<u>Hint</u>: You may find the following relationship useful: $\dfrac{d\ln f(x)}{dx} = \dfrac{f'(x)}{f(x)}$

4. An element of wire of oriented length $d\vec{l}$ is moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$.

(a) (3 pts) Show that the motional EMF across the length of this element is

$$d\Phi = (\frac{\vec{v}}{c} \times \vec{B}) \cdot d\vec{l}$$

(b) (7 pts) A conducting spherical shell of radius R rotates with its axis aligned with the z-axis at an angular velocity ω. The sphere is in a constant magnetic field $\vec{B} = B_0 \hat{k}$ also in the z-direction. Find the electric potential of the surface of the sphere, $\Phi(\theta)$ where θ is the usual polar angle.

5. Consider a system consisting of very large number $N \to \infty$ of non-interacting *distinguishable* particles at rest. Each particle has only two nondegenerate energy levels, 0 and ε.

a. (3 pts) What is the mean energy u per particle in the two limits $T \to 0$, and $T \to \infty$? Justify your answer.

b. (3 pts) What is the entropy per particle in the two limits $T \to 0$, and $T \to \infty$? Justify your answer.

c. (3 pts) Compute the mean energy per particle in this system at an arbitrary temperature. At what (positive) temperature does this mean energy attain maximum? Compare it with the appropriate limiting case of part a).

d. (3 pts) Compute the entropy per particle at an arbitrary temperature. Compare it with the limiting cases of part b).

e. (3 pts) If you are also allowed to consider negative temperatures, at what temperature does the mean energy attain maximum? What is this energy?

6. Consider an infinite square well in one dimension (x), with $V(x) = \alpha$, a positive constant for $0 < x < L$, and $V(x) = \infty$ otherwise.

a. (3 pts) For $0 < x < L$ derive, from the time-dependent Schrödinger equation, the differential equation that determines the wavefunctions of the states of definite energy. Then find the most general solutions of that equation.

b. (3 pts) Use the boundary conditions to find the complete set of states of definite energy indicating the energy of each one. Show that no solution exists for $E < \alpha$.

c. (3 pts) Suppose that the wavefunction at $t = 0$ is

$$\Psi(x,0) = \frac{N}{\sqrt{L}}\left(\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}\right).$$

Find $\Psi(x;t)$ for any later time t.

d. (3 pts) Is $\Psi(x,0)$ even, odd, or neither about the point $x = L/2$? Prove your answer. [It might be easier to determine this by shifting the variable to $y = x - (L/2)$.]

e. (3 pts) Show how an 'even' or an 'odd' answer to (d) would immediately determine the expectation value of x.