Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

FRIDAY, January 3, 2013
9:00 AM — 1:00 PM

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of six problems, each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. Your special ID number that you received from Delores Cowen;
2. The problem number and the title of the exam (*i.e.* Problem 1, Part I).

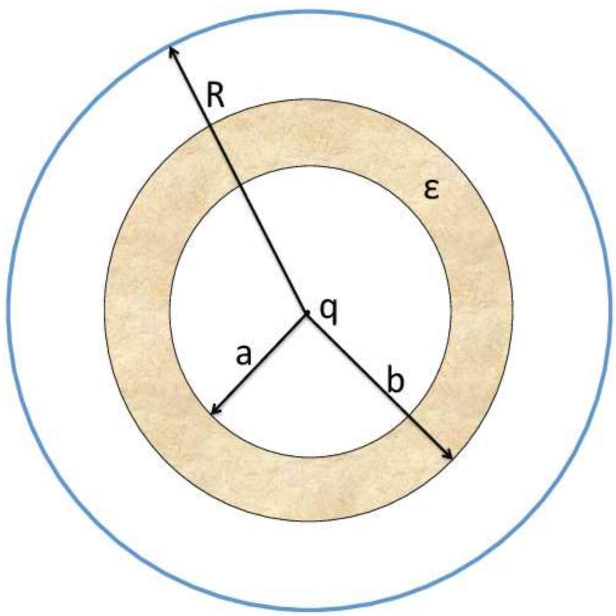
Please make sure your answers are dark and legible.

Do NOT write your name on the cover or anywhere else in the booklet!

1. (**10 points**) A point charge $+q$ is surrounded by a dielectric spherical shell with the inner and outer radii a and b respectively, which in turn is surrounded by an infinitesimally thin conducting shell of radius R. Both the dielectric and conducting shells are concentric with the location of the point charge (see the figure below). The dielectric has a homogeneous (scalar) permittivity ϵ.

   (a) Find the potential and electric field everywhere. (6 pts)

   (b) Find the surface charge density on all surfaces. (4 pts)



2. (**10 points**) A particle of mass m is constrained to move on a spherical surface of radius R, subject to a potential $U = m\mathbf{A} \cdot \mathbf{r}$, where $\mathbf{A}$ is a vector of suitable dimension with magnitude A and direction (θ_A, φ_A).

(a) Derive the particle Lagrangian. (3 pts)

(b) Derive the equations of motion. (4 pts)

(c) Now assume that the $\mathbf{A}$ vector points along the z-axis, and the particle has velocity and location in the x-y plane at $t = 0$. Describe the initial motion. (3 pts)

3. (**10 points**) Consider two spin-1/2 particles.

(a) Initially these two particles are in a spin singlet state. If a measurement shows that particle 1 is in the eigenstate of $S_x = -\hbar/2$, what is the probability that particle 2 in this same measurement is in the eigenstate of $S_z = +\hbar/2$? (4 pts)

(b) If initially particle 1 is in a state given by $a_1\chi_+ + b_1 e^{i\alpha_1}\chi_-$ and particle 2 is in a state given by $a_2\chi_+ + b_2 e^{i\alpha_2}\chi_-$, what is the probability that after a measurement these two particles are in a spin triplet state? Here χ_+ and χ_- are the standard eigenvectors (eigenspinors) of spin operator $\hat{S}_z$ for a spin-1/2 particle, and a_i, b_i, α_i $(i = 1, 2)$ are real constants. (6 pts)

Hint: The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4. (**10 points**)

(a) Derive the Clausius-Clapeyron relation for the coexistence of two phases (e.g., liquid-gas or liquid-solid). This is a relation between dP/dT (with pressure P and temperature T), specific latent heat L and the specific volume difference of the two phases $(v_1 - v_2)$ of a single constituent. (5 pts)

(Hint: Possible methods of the derivation include using the Carnot cycle or making use of chemical potentials.)

(b) A long vertical cylindrical column consisting of a substance is initially at a temperature T in a gravitational field g. Above a certain point along the column the substance is in liquid state, and below it is solid. When the temperature is lowered by ΔT, the position of the solid-liquid interface moves upward by a distance h. Neglecting the thermal expansion of the solid, find the density ρ_L of the liquid in terms of the density ρ_S of the solid. The specific latent heat of the solid-liquid phase transition is L. (5 pts)

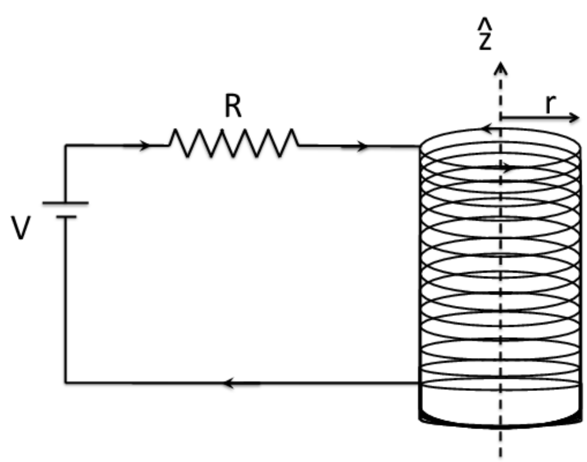
5. (**10 points**) A long solenoid of length L and radius r with N closely spaced turns is placed with its axis along the z direction. It is connected to a battery with negligible internal resistance and a voltage V. The current enters the top of the solenoid and exits through the bottom; looking from above this current is along the anti-clockwise direction. A resistance R is connected to the battery in series with the solenoid, and the resistance of the solenoid is negligible.

(a) Use Ampere's law to find the magnetic field B inside the solenoid when a current I is flowing through it. (2 pts)

(b) Due to the self-inductance of the solenoid, the current I will take some time to reach its final value. At an intermediate time t, derive a formal expression for the electric field at an arbitrary radial distance ρ inside the solenoid as a function of the current change rate dI/dt. Express your result in both Cartesian coordinates and Cylindrical coordinates. (3 pts)

(c) Using the formula derived above and setting $\rho = r$, find the potential change across one turn of the solenoid. Summing over all the turns, deduce the self-inductance of the solenoid. (3 pts)

(d) Given the formula for the inductance and the result of (b), find the time dependent form for the electric field at a radial distance ρ within the solenoid as a function of the dimensions of the solenoid and the resistance of the entire circuit. (2 pts)



6. (**10 points**) For a one-dimensional simple harmonic oscillator with potential $V(x) = m\omega^2 x^2/2$, it is known that the ground state is described by $\psi(x) = A\exp(-\frac{m\omega}{2\hbar}x^2)$ if the nonrelativistic kinetic energy $K = p^2/2m$ is used.

(a) Determine A. (2 pts)

(b) Prove that if using relativistic kinetic energy, the lowest order correction to K is given by $-p^4/(8m^3c^2)$, where c is the speed of light. (3 pts)

(c) Use the result of (a) and perturbation theory to calculate the ground state energy of this relativistic harmonic oscillator up to order $1/c^2$. (5 pts)

Ph.D. QUALIFYING EXAMINATION
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WAYNE STATE UNIVERSITY

PART II

MONDAY, January 6, 2013
9:00 AM — 1:00 PM

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of six problems, each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. Your special ID number that you received from Delores Cowen;
2. The problem number and the title of the exam (*i.e.* Problem 1, Part II).

Please make sure your answers are dark and legible.

Do NOT write your name on the cover or anywhere else in the booklet!

1. (**10 points**) Consider a system of a large number of distinguishable atoms N which are always at rest and have three non-degenerate energy levels, $-\mathcal{E}$, 0, and $+\mathcal{E}$. The system is in contact with a thermal reservoir at temperature T.

   (a) Compute the partition function for this system of N particles. (2 pts)

   (b) Compute the average internal energy per atom. (2 pts)

   (c) What is the average internal energy per atom in the limit of $T \to 0$ and $T \to \infty$? (2 pts)

   (d) Calculate the entropy per atom. (2 pts)

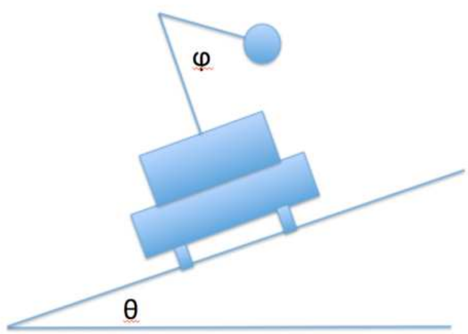
   (e) What is the entropy per atom in the limit of $T \to 0$ and $T \to \infty$? (2 pts)

2. (**10 points**) A car makes a turn on a road tilted by an angle θ. Inside the car there is a pendulum, which during the turn moves to an angle φ with respect to its support (see the figure).

(a) Evaluate the angle between the pendulum and the vertical. (1 pt)

(b) Evaluate the force of friction as a function of the given angles and the car weight W. (4 pts)

(c) Evaluate the coefficient of static friction between the car and the road, if the car is barely able to complete the turn without skidding. (5 pts)



3. (**10 points**) A particle of mass m and charge q hangs from an ideal spring with spring constant k. It is displaced by a distance z_0 from rest and set into a state of small oscillations (along the vertical z-axis). The dipole electric and magnetic fields of a changing charge distribution at r, θ, ϕ (in spherical coordinates) are approximated as

$$\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin\theta}{r} \cos\left[\omega(t - r/c)\right] \hat{\theta}, \quad \vec{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi c} \frac{\sin\theta}{r} \cos\left[\omega(t - r/c)\right] \hat{\phi},$$

where p_0 is the maximum dipole moment of the charge distribution at a given time t, ω is the frequency of small oscillations, r is the distance from the center of the dipole, μ_0 is the permeability of vacuum, and c is the speed of light in vacuum.

(a) Evaluate the Poynting vector as a function of the solid angle. (3 pts)

(b) Averaging over a cycle, evaluate the total power emitted. (4 pts)

(c) Using the result of (b), evaluate the time dependence of the maximum displacement of the particle $z_0(t)$. (3 pts)

4. (**10 points**) The Helmholtz free energy of a photon gas at temperature T inside a container of volume V is $F = -\alpha V T^4$, where α is a constant.

   (a) Using the above information, calculate the internal energy E of a photon gas and derive the relationship between E and PV, where P is the pressure of the gas. (3 pts)

   (b) Derive the average number of photons $\bar{N}_{ph}$ in a volume V at temperature T. You do not need to evaluate any integrals. (5 pts)

   (c) Using the results from (a) and (b), derive the equation of state for a photon gas in terms of P, V, and $\bar{N}_{ph}$ and compare it to the equation of state for a classical ideal gas. (2 pts)

5. (**10 points**) A particle is subject to a central force $F(r) = -k/r^{\alpha}$, where r is the radius of the particle orbit and k is a constant.

(a) Prove that the orbit must be circular if the particle energy is equal to its effective potential energy $V(r) = U(r) + l^2/(2mr^2)$, where l is the particle angular momentum and m is the particle mass. Find the value of the orbit radius r_0. (3 pts)

(b) Evaluate for which range of α the circular orbit is stable. (4 pts)

(c) Within the α range that you found, compute the frequency of small radial oscillations around the nominal circular orbit, when the particle is perturbed by a small radial displacement. (3 pts)

6. (**10 points**) The wavefunction of a particle is given by $\psi = A(x + 2z)\exp(-\alpha r)$, where $r = \sqrt{x^2 + y^2 + z^2}$ and α is a real constant.

(a) Find A. (3 pts)

(b) What are the expectation values of the orbital angular momentum operator L^2 and the z-component angular momentum L_z? (5 pts)

(c) If measuring L_z, what is the probability of getting a value of $+\hbar$? (2 pts)

Hint: The first few spherical harmonics are given by

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}, \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta, \quad Y_1^{\pm 1} = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi},$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1), \quad Y_2^{\pm 1} = \mp \left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi}.$$