Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

FRIDAY, May 2, 2014
9:00 AM — 1:00 PM

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of six problems, each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. Your special ID number that you received from Delores Cowen;
2. The problem number and the title of the exam (*i.e.* Problem 1, Part I).

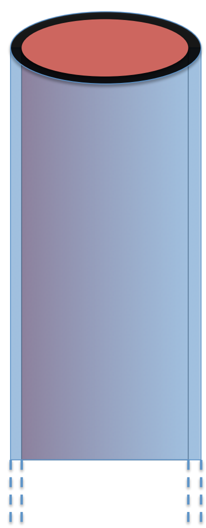
Please make sure your answers are dark and legible.

Do NOT write your name on the cover or anywhere else in the booklet!

1. (**10 points**) Two very long cylinders with radii R and $R + \delta R$ ($\delta R \ll R$) are placed concentrically (one inside the other) and stood upright as shown in the figure below. Both cylinders are made of insulating material with permittivity ϵ and permeability of vacuum. The inner cylinder is solid while the outer one is a shell. Positively charged particles are then filled and moved inside the narrow cylindrical shell along the edge in a cyclonic velocity pattern $\vec{v} = v_z\hat{k} + v_\phi\hat{\phi}$. These particles are moved by non-electromagnetic forces and have a volume charge density ρ. The gap between the cylinders is narrow enough that one can approximate this shell as an effective charged surface.

   (a) Calculate the effective surface charge density and current density. Ignore any Lorentz force or time dependent effects. (4 pts)

   (b) Calculate the E and B field everywhere for this configuration. Ignore any Lorentz force or time dependent effects. (6 pts)



2. (**10 points**) Two particles of masses m_1 and m_2 are initially at rest and separated by a distance a. They fall towards each other due to a force $-k/r^2$. Use the reduced mass to solve the followings:

   (a) Calculate the energy of the system. (2 pts)

   (b) From the system energy, derive an equation for the time t_0 that it takes for the two particles to fall onto each other. (5 pts)

   (c) Solve the equation to get t_0, and compare it to the orbital period t_1 if the two particles were orbiting each other in a circular orbit. (3 pts)

   (Hint: You might need to use the change of variable $u^2 = 1 - r/a$ to perform the integration.)

3. (**10 points**) Consider a particle of mass m in a one-dimensional potential

$$V(x) = \begin{cases} 0, & 0 \le x \le a, \\ \infty, & x < 0 \text{ or } x > a. \end{cases}$$

At $t = 0$ the particle's initial wave function is given by

$$\Psi(x, t = 0) = \sqrt{\frac{32}{5a}} \sin(2\pi x/a) \cos^2(\pi x/a).$$

(a) What is the expectation value of the particle energy at $t = 0$? (4 pts)

(b) What is the wave function at a later time $t = T$? What is the expectation value of the particle energy at this time? (3 pts)

(c) What is the probability that the particle is found in the left half of the potential well (i.e., $0 \le x \le a/2$) at $t = T$? Show your calculation steps. (3 pts)

4. (**10 points**) A cylindrical rod of length l insulated on the lateral surface is initially in contact at one end with a reservoir at temperature $T_H = 400$ K and the other end with another reservoir at a lower temperature $T_C = 300$ K. The temperature within the rod initially varies linearly with position x according to $T_i(x) = T_H - (\frac{T_H - T_C}{l})x$. The rod is then isolated from the two reservoirs simultaneously and eventually it comes to a final equilibrium at a temperature T_f. The mass and the specific heat of the rod are $m = 1$ kg and $c_p = 385$ J/(kg·K) respectively.

   (a) Considering the rod thermally isolated after being disconnected from the reservoirs, express T_f in terms of T_H and T_C. (1 pt)

   (b) For a segment dx located at x, evaluate the entropy change dS from the moment of disconnecting the reservoirs to the moment of reaching T_f. (3 pts)

   (c) Evaluate the entropy change for the whole rod from the moment of disconnecting the reservoirs to the moment of reaching T_f. (3 pts)

   (d) If only the cold reservoir is removed instead and an equilibrium is reached, evaluate the entropy change in the rod. (3 pts)

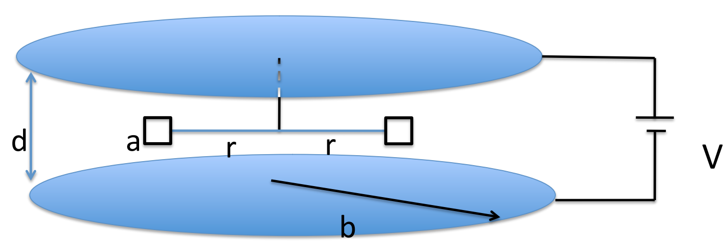
5. (**10 points**) Consider a circular-shaped, parallel plate capacitor which is being slowly charged (see the figure below). Two conducting square-shaped wire loops (with side a and a small resistance Ω_2 for each side) are attached to an insulating rod of length $2r$. The rod is hung at the middle from the center of the top capacitor plate by an insulating thread. The entire hanging assembly is free to move and rotate. The rod can expand or contract lengthwise with an elastic restoring force $F = -k\Delta r$. Assume the maximum voltage of the battery to be V.

(a) Assuming that the hanging bracket assembly within the capacitor has a minimal effect on the overall charging of the capacitor, calculate the potential drop across the capacitor plates as a function of time t, maximum voltage V, the resistance in the outer circuit R and the capacitance C. (2 pts)

(b) Use this time-dependent potential to find the time-dependent electric field E and the time-dependent magnetic field B in the capacitor (again ignoring the effect of the hanging assembly within). (3 pts)

(c) From the expression of the space-time dependence of the B field, calculate the current in the square loops and hence the net Lorentz force on the squares when they are at a distance r from the center. Ignore any transient currents from the E field. (3 pts)

(d) Quantitatively describe what happens to the hanging bracket assembly as the capacitor begins to charge up. (2 pts)



6. (**10 points**)

(a) Prove the three-dimensional virial theorem $2\langle K \rangle = \langle \mathbf{r} \cdot \boldsymbol{\nabla} V \rangle$ for any stationary state of a nonrelativistic particle, where K is the kinetic energy and V is the potential energy. (6 pts)

(Hint: use the relation $d\langle Q \rangle/dt = (i/\hbar)\langle [\hat{H}, \hat{Q}] \rangle + \langle \partial \hat{Q}/\partial t \rangle$ and set $Q = \mathbf{r} \cdot \mathbf{p}$.)

(b) Based on the result of (a), show that for a nonrelativistic particle of mass m in a logarithmic central potential $V(r) = \alpha \ln(r/r_0)$, where α and r_0 are constants and $r = |\mathbf{r}|$, all eigenstates have the same mean-squared velocity $\langle v^2 \rangle$. Find this mean-squared velocity. (4 pts)

Ph.D. QUALIFYING EXAMINATION
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PART II

MONDAY, May 5, 2014
9:00 AM — 1:00 PM

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of six problems, each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. Your special ID number that you received from Delores Cowen;
2. The problem number and the title of the exam (*i.e.* Problem 1, Part II).

Please make sure your answers are dark and legible.

Do NOT write your name on the cover or anywhere else in the booklet!

1. (**10 points**) A very tall cylinder with a cross-section of 1 m^2 is set vertically from the ground. The air temperature inside the tube is uniformly 300 K. The air particle density at the ground level is $n_0 = 2.69 \times 10^{25}$ m^{-3}, and the molar mass of air is $M = 29$ g/mol. Assume ideal gas law for the air.

   (a) Calculate the air pressure as a function of height Z. (7 pts)

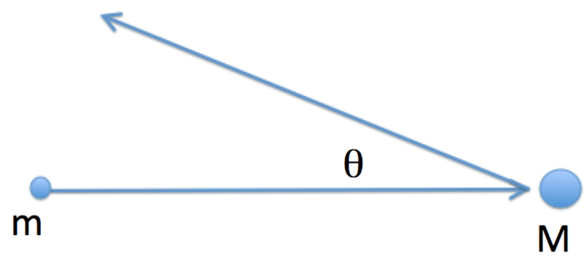
   (b) Find the total number of air particles inside the tube. (3 pts)

2. (**10 points**) In Rutherford back-scattering, atoms are identified by shooting an α particle of mass m and initial kinetic energy E onto a rest target of unknown mass M. The particle is then detected at an angle θ from the back-scattering direction, with measured energy E_1.

   (a) Write down the energy and momentum conservation equations. (2 pts)

   (b) Solve for M as a function of the given quantities. (5 pts)

   (c) If $m = 4$ atomic units (a.u.), $E = 2$ MeV, $E_1 = 0.71$ MeV, and $\theta = 10\ mrad$, find which atom is being bombarded. (3 pts)



3. (**10 points**) Positronium is a bound state of an electron and a positron. Both the energy levels and wave functions of positronium are similar to those of hydrogen.

(a) What is the normalized wave function of the $1s$ ground state for positronium (without considering spin)? Use the result of $1s$ ground state for hydrogen: $\psi_{100} = \exp(-r/a_0)/\sqrt{\pi a_0^3}$, where the Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/(m_e e^2)$ with m_e the electron mass. (2 pts)

(Hint: use the reduced mass.)

(b) In positronium the spin operators of electron and positron are given by $\hat{\mathbf{S}}_e$ and $\hat{\mathbf{S}}_p$. What are the allowed values of quantum numbers for the total spin S and the corresponding z-component S_z? (2 pts)

(c) The $1s$ state of positronium is subjected to a hyperfine interaction

$$\hat{H}' = -\beta\hat{\boldsymbol{\mu}}_e \cdot \hat{\boldsymbol{\mu}}_p \delta(\mathbf{r}),$$

where $\hat{\boldsymbol{\mu}}_e = -\frac{e}{m_e c}\hat{\mathbf{S}}_e$ and $\hat{\boldsymbol{\mu}}_p = \frac{e}{m_e c}\hat{\mathbf{S}}_p$ are the magnetic moments and β is a constant. Use first-order perturbation theory to calculate the change of system energy. You need to consider each spin state obtained in (b). (6 pts)

4. (**10 points**) A two-level system of $N = n_1 + n_2$ particles is distributed among two eigenstates 1 and 2 with eigenenergies E_1 and E_2 respectively. The system is in contact with a heat reservoir at temperature T. If a single quantum emission into the reservoir occurs, population changes $n_2 \to n_2 - 1$ and $n_1 \to n_1 + 1$ take place in the system.

   (a) Give Boltzmann's statistical definition of entropy and present its physical meaning briefly but clearly. (1 pt)

   (b) For $n_1 \gg 1$ and $n_2 \gg 1$ and using the result of (a), obtain the expression for the entropy change of the two-level system in terms of n_1 and n_2. (3 pts)

   (c) Find the entropy change of the reservoir. (3 pts)

   (d) From results of (a) and (b), derive the Boltzmann relation for the ratio n_2/n_1. (3 pts)

5. (**10 points**) A particle of mass m moves on a paraboloid $z = \alpha(x^2 + y^2)$, subjected to the force of gravity mg directed towards the negative z direction.

   (a) Derive the particle Lagrangian. (5 pts)

   (b) Is angular momentum conserved in this system? Discuss. (2 pts)

   (c) Find the condition for which the particle follows a circular motion. (3 pts)

6. (**10 points**) Consider a sphere with radius R made of a linear homogeneous dielectric material with permittivity ϵ, concentrically surrounding a solid conducting sphere of radius a (see the figure below). Charges are distributed (glued) on the outer dielectric surface (i.e., the spherical surface of radius R) with density $\sigma = \sigma_0 + \sigma_1 \cos\theta$, where θ is the polar angle.

(a) Find the electric field E and potential V of this configuration at a distance $r \gg R$. Use simple physical arguments and approximations. Don't solve Laplace equation for this part. (3 pts)

(b) Set up Laplace equation for the potential in spherical coordinates and specialize to the case of azimuthal-angle ϕ independent solutions. Introduce separation of variables and write down the most general form of the solution which has no ϕ dependence. Write down all the boundary conditions on the potential and the electric field that are needed to find the potential everywhere in this problem. Here you may express the induced charge density on the conductor as $\sigma_c(\theta)$. (3 pts)

(c) Use the results of (b) to find the potential V everywhere for the given configuration. (4 pts)