Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I: CLASSICAL MECHANICS

Wednesday, May 4, 2016
10 AM — 12 noon

Room 245, Physics Research Building

INSTRUCTIONS: This examination consists of three problems each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. your special ID number that you received from Delores Cowen,
2. the exam and problem number (*i.e.* I.1).

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Problem I.1. Meteor Motion in the Earth's Atmosphere.

Consider a meteor in the Earth's atmosphere moving parallel to the Earth's surface at a low altitude - an altitude that is small compared to the Earth's radius. The meteor loses momentum as it collides with air molecules. In a simple model, the meteor can be treated as a circular disk that pushes the initially stationary air molecules so that they acquire the velocity and direction of the meteor at the time they are pushed. Assume that the downward velocity of the meteor is much smaller than its horizontal velocity and hence the downward motion can be ignored. The meteor has cylindrical radius R and mass M. At the altitude of the meteor, the atmosphere has volume mass density ρ.

a) (4 pts.) Find the momentum dp lost to the air in time dt.

b) (3 pts.) Using the result of a), find a differential equation relating the acceleration of the meteor and its instantaneous velocity.

c) (3 pts.) Solve the equation to find the time required to reduce the velocity by half.

Note of interest: The mass of a meteor can be estimated by observation of its instantaneous velocity and acceleration as found in b). The mass thus estimated is known as the "dynamical mass" and is accurate to roughly a factor of two.

Problem I.2.

Consider two identical beads of mass m, each carrying a charge q, constrained to move without friction on a horizontal wire ring of radius a.

a) (6 pts.) Find the Lagrangian and derive the equations of motion.

b) (2 pts.) Find a stationary solution.

c) (2 pts.) Consider a small perturbation to the stationary state and find the frequency of small oscillations.

Problem I.3.

Consider two particles each of mass m and charges $q_1 = -q_2 = q > 0$ that are connected by a spring of spring constant k. The particles and spring are located on a horizontal frictionless surface. The system is immersed in a strong uniform magnetic field of strength B oriented perpendicular to the surface. Neglect the electrostatic interaction of the particles between themselves and take into account only the interaction with the magnetic field and the spring. Use a Cartesian coordinate system with the magnetic field pointing along the positive z-direction. The particles are initially located at positions $(a, 0, 0)$ and $(-a, 0, 0)$. The particles are initially at rest with the spring stretched such that $2a > L_0$, where L_0 is the length of the unstretched spring.

a) (3 pts.) Draw a diagram indicating all forces acting on the particles (after they start to move) and show that the trajectory of the particles are symmetric with respect to the $y - z$ plane.

b) (4 pts.) Find the equations of motion (you might want to consider only the positively charged particle) and qualitatively describe the motion of the system.

c) (3 pts.) Solve the equations and find the frequency of oscillations if any.

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PART II: ELECTRICITY AND MAGNETISM

Wednesday, May 4, 2016
1:30 PM — 3:30 PM

Room 245, Physics Research Building

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Problem II.1.

A point charge q is held at a point $D\hat{z}$ near a thin grounded infinite conducting plate, which occupies the x-y plane. Ignore gravity in this problem.

a) (1 pt.) Explain whether the charge feels an attractive or repulsive force with respect to the plate? HINT: use the method of images.

b) (3 pts.) Find the electric field in the positive-z and negative-z hemispheres, above and below the plate. Find the charge density on the plate.

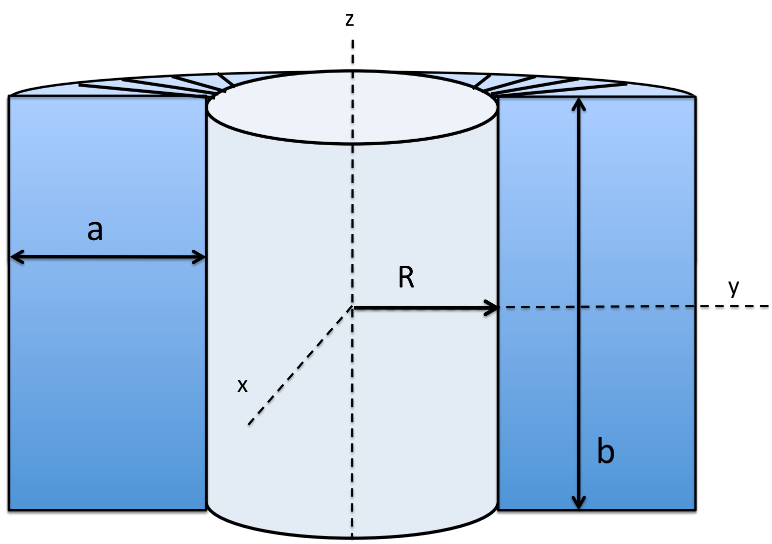
c) (2 pts.) Now consider another positive charge q placed at the point $-D\hat{z}$, the location of the first image charge. Make a sketch of the electric field in both hemispheres above and below the infinite grounded conducting plate.

d) (2 pts.) Find the charge density on the infinite conducting plate.

e) (2 pts.) Explain whether the net force experienced by the two charges is such that they are attracted towards each other and the plate, or they are repelled from each other and the plate. Find the magnitude of this force.

Problem II.2.

A toroidal inductor with inner radius R, has N turns of wire wound over a frame with a rectangular cross-section with a shorter side a and a longer side b, as shown in the Figure. The cylindrical space of radius R and height b, as well as the space within the toroid is filled with air, i.e., $\epsilon = \epsilon_0$, $\mu = \mu_0$.



a) (2 pts.) Assuming a steady current I through the coils, derive an expression for the B field within the toroid as a function of the radial distance ρ from the central z-axis and as a function of z. The origin of the coordinate system is at the center of the toroid.

b) (2 pts.) Find the self inductance L of this toroid. Assume all materials used to make the toroid have $\epsilon = \epsilon_0$ and $\mu = \mu_0$).

c) (2 pts.) Assume that this toroid is connected to a battery with voltage V_0 and resistance R, and that the resistance of the toroid is negligible compared to R. Find the potential drop as a function of time t across the inductor in terms of the L calculated in part b).

d) (4 pts.) Assuming that $b >> a$, find a general form for the induced electric field at time t at a point $(x, y, 0)$ inside the toroid. Use the two Maxwell equations $\nabla \cdot E = 0$ and $\nabla \times E = -\frac{\partial B}{\partial t}$. There is no need to evaluate any constants in the expression. A table of vector derivatives follows.

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{\imath} + \frac{\partial t}{\partial y}\hat{\jmath} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl. $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) + \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

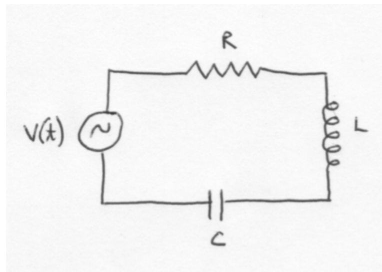
Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Problem II.3. Series RLC Circuit

Consider an alternating voltage source connected to a series R-L-C circuit as shown in the Figure. The voltage source provides a time-dependent voltage given by $V(t) = V_0 \sin(\omega t)$ where V_0 and ω are real, positive numbers and t is time.



a) (4 pts.) Using Kirchoff's Law find a differential equation for the current $I(t)$ in terms of V_0, ω, t, R, L, and C.

b) (4 pts.) Assuming a solution of the form $I(t) = I_0 \sin(\omega t + \phi)$, find I_0 and ϕ. Hint: use trigonometric identities so that the arguments of trigonometric functions are the same.

c) (2 pts.) Find the resonant frequency ω_0.

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PART III: QUANTUM MECHANICS

Friday, January 8, 2016
10 AM — 12 noon

Room 245, Physics Research Building

INSTRUCTIONS: This examination consists of three problems each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

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Problem III.1. A beam of particles can be described by a quantum-mechanical wave. Consider the 1-dimensional motion of a beam of particles of mass m and energy $E > 0$ traveling (non-relativistically) in the $-x$ direction and incident on a step potential at $x = 0$. (The particles have $x > 0$ before they reach the step.) The potential energy is described by

$$U(x) = \begin{cases} -U_0 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

where U_0 is a positive real number.

a) (2 pts.) State the time-independent Schroedinger equation for each region of x.

b) (2 pts.) State the form of the solutions for each region of x.

c) (2 pts.) State the boundary conditions.

d) (3 pts.) Find the probability that a particle is back-scattered (reflected).

e) (1 pt.) Using the result in d) find the probability for back scattering in the classical limit.

Problem III.2. A one-dimensional quantum harmonic oscillator of mass m has ground state time-independent wavefunction $\psi_0(x)$ for potential energy $V(x) = \frac{1}{2}kx^2$ $(k > 0)$.

a) (6 pts.) Find $\psi_0(x)$ with the correct normalization factor using the lowering operator $\hat{a}$ for which $\hat{a}\psi_n(x) = \sqrt{n}\psi_{n-1}(x)$, where $\hat{a} = i\hat{p} + m\omega x$, $\omega = \sqrt{k/m}$, and $\hat{p}$ is the momentum operator. A mathematical table follows.

b) (4 pts.) At a certain point in time, k is instantly doubled so that the potential enmergy becomes $V_0' = kx^2$ for which the ground state is $\psi_0'(x)$. For the instant when V_0 is changed into V_0', find the numerical value of the probability of finding the particle in state ψ_0'.

Mathematical Formulas

Trigonometry:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Integrals:

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Exponential integrals:

$$\int_0^\infty x^n e^{-x/a}\,dx = n!\,a^{n+1}$$

Gaussian integrals:

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\,dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2}\,dx = \frac{n!}{2}a^{2n+2}$$

Integration by parts:

$$\int_a^b f\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}g\,dx + fg\Big|_a^b$$

Problem III.3. A spin 1/2 particle interacts with a magnetic field $\vec{B} = B_0\hat{z}$ through the Pauli interaction $H = \mu\vec{\sigma} \cdot \vec{B}$ where μ is the magnetic moment.

The Pauli spin matrices are $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ where the σ_i are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenstates for σ_z are the spinors

$$\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) (3 pts.) Suppose that at time $t = 0$ the particle is in an eigenstate χ_x corresponding to spin pointing along the positive x-axis. Find the eigenstate χ_x in terms of α and β.

(b) (7 pts.) For a later time t, find the probability that the particle is in an eigenstate corresponding to the spin pointing along the negative y-axis.

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PART IV: THERMAL, STATISTICAL, AND MODERN PHYSICS

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1:30 PM — 3:30 PM

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Problem IV.1. One mole of a van-der-Waals gas (at pressure P_i and temperature T_i) is held within a container with a movable piston. The walls of the container are adiabatic (cannot absorb any heat). A piston can be used to change the volume V of the gas in the container. There is a valve on the piston that when opened allows gas to freely flow through it. Initially the gas is only in the bottom portion of the container with a volume $V_0/2$.

a) (5 pts.) The valve is opened and the gas is allowed to expand into the remainder of the container (the full volume V_0). Assuming that the specific heat at constant volume c_V is independent of temperature, what is the new temperature of the gas?

b) (5 pts.) Then the piston is drawn completely to the top. The valve is now shut and the piston is pushed down until the entire gas is in the bottom of the container within volume $V_0/2$. What is the new temperature of the gas ?

The van-der-Waals gas has an equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \tag{1}$$

where a, b are small constants.

Maxwell's relation is

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T . \tag{2}$$

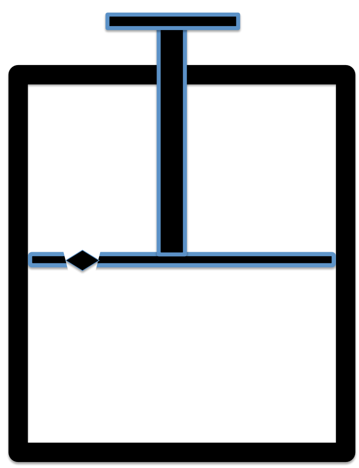

Figure 1: A gas is contained in a volume with a movable piston and a valve.

Problem IV.2. A box contains 4 distinguishable (classical) particles and 4 energy levels $0, a, 2a$ and $3a$ (where a is a positive constant). Ignore zero point energy in all calculations below.

a) (3 pts.) Initially the box is completely thermally insulated, with total energy $4a$ inside. What is the degeneracy of this state ?

b) (3 pts.) In the next scenario, the box is held in thermal contact with a heat reservoir at a temperature $T = 1/\beta$ (Canonical Ensemble). What is the partition function for the box? What is the mean energy at a temperature T of the reservoir? Find the temperature T_{4a} at which the mean energy of the box is equal to $4a$. Just set up the equation for this temperature, but do not solve it.

c) (4 pts.) Now, consider the case where the box in part a) is brought in contact with a reservoir at a temperature of T_{4a}. What is the mean energy transferred between the reservoir and the box? What is the variance (fluctuation) of this energy transfer?

Problem IV.3. A photon of energy E collides with a stationary electron whose rest mass energy is 511 keV. After the collsion the photon scatters at an angle θ (with respect to its original direction) with energy E'. The motion of the electron can be described classically (non-relativistically).

Answer the following questions in terms of the given variables and fundamental constants.

a) (1 pt.) What is the initial wavelength of the photon?

b) (2 pts.) What is the final-state (DeBroglie) wavelength of the electron?

c) (2 pts.) Suppose that the photon loses 100 eV of energy in the collision. Find the factor $\beta = v/c$ of the final state electron.

d) (5 pts.) Denote the ratio of E to electron rest mass energy by r and the ratio of E' to E by f. Find the scattering angle of the *electron* relative to the initial photon direction.