Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

Friday, May 3, 2019
10:00 — 12:00

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of three problems each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. your special ID number that you received from Delores Cowen,
2. the problem number (*i.e.* Problem 7).

Please make sure your answers are dark and legible.

Do NOT write your name on the cover or anywhere else in the booklet!

1. **(10 points)** A planet of mass m orbits a star of mass M. The orbit is elliptic and the distance between the planet and the star oscillates between r_0 and r_1. The system total energy is E and its total angular momentum is L.

   a) (1 pt) Write the reduced mass μ of the system .

   b) (4 pts) Write an energy conservation equation for the system energy E as a function of r, its time derivative, and the angular momentum L.

   c) (3 pts) Solve this equation for r_0 and r_1.

   d) (2 pts) Using these solutions, solve for L and E as a function of the masses, the gravitational constant G, r_0 and r_1.

2. **(10 points)** A particle of mass m moves without friction on a paraboloid $z = \alpha(x^2 + y^2)$, subjected to the force of gravity mg directed towards the negative z direction.

a) (3 pts) Derive the particle's Lagrangian.

b) (1 pts) Is angular momentum conserved in this system? Discuss.

c) (2 pts) Find the conditions (radius and angular velocity) for which the particle follows a circular motion.

d) (4 pts) Calculate the frequency of small oscillations near a circular orbit.

3. **(10 points)** A spherical interstellar grain has a radius of 1 μm and a density of $\rho = 3$ g/cm^3. Consider such a grain at the distance of 1 A.U. (1.5×10^8 km) and assume it completely absorbs the light from the Sun. Considering the total radiation intensity of the Sun is 1400 W/m^2 at 1 A.U..

   a) (4 pts) Calculate the force on the grain due to radiation pressure.

   b) (4 pts) Calculate the force on the grain due to the Sun's gravity.

   c) (1 pt) Determine the direction of motion of the grain.

   d) (1 pt) Determine the acceleration of the grain.

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PART II

FRIDAY,May 3, 2019
13:30 — 15:30

ROOM 245 PHYSICS RESEARCH BUILDING

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4. **(10 points)** A conducting sphere of radius R has net charge Q. It also has uniform magnetization $\mathbf{M}$. Formulae to transform spherical and cartesian coordinates can be found in the Math book.

   a) (1pt.) Derive the electric field $\mathbf{E}$.

   b) (1pt.) Derive the total magnetic dipole $\mathbf{m}$.

   c) (4pts.) Derive the magnetic field $\mathbf{B}$ outside the sphere.

   d) (4pts.) Make a choice of axes and derive the total momentum $\mathbf{P}$ of the configuration.

5. **(10 points)** A capacitor is made of concentric spherical shells of radii a, b, with $b > a$. The volume between the shells is filled with a weakly conducting medium of conductivity $\sigma = \sigma_0 a/r$, where σ_0 is constant and r is the distance to the common center. A battery is connected so that the inner shell is at potential V_0 and the outer shell is at ground. Current flows through the medium.

   a) (4 pts) Find the total resistance R of the medium.

   b) (2 pts) Find the free charge inside an arbitrary radius r within the medium.

   c) (2 pts) Find the total capacitance C of the configuration.

   d) (2 pts) If the battery is removed what will be the characteristic decay time of the current?

6. **(10 points)** A dipole $\mathbf{p}$ is at a distance $z = d$ from the infinite conducting grounded $(x - y)$ plane. It is oriented so as to make an angle θ with the $z-$axis.

   a) (4 pts) Find the potential everywhere.

   b) (4 pts) Find the torque on the dipole

   c) (2 pts) If there is a torque, at which position(s) it is zero and is the equilibrium stable or unstable?

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PART III

MONDAY, May 6, 2019
10:00 — 12:00

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of three problems each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

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Problem 7 (10 points)

N molecules of a gas are in a thermally isolated vessel of volume V_i. The initial temperature of this gas is T_i. A door is opened and the gas is allowed to abruptly and freely expand into a total volume V_f. No work is done and no heat is transferred to the gas. After a long time, the gas reaches a final temperature T_f.

(a) (2 pts) Consider the case that the gas is ideal. What is the change in internal energy E as a result of this expansion?
(b) (2 pts) What is the change in entropy of the gas as a result of the expansion? Is the process reversible or irreversible?

(c) (2 pts) Consider a non-ideal gas. Show that the temperature change can be expressed as

$$\left(\frac{\partial T}{\partial V}\right)_E = -\frac{1}{C_V}\left[T\left(\frac{\partial S}{\partial V}\right)_T - p\right]$$

Note the Maxwell relation: $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$

(c) (4 pts) In the case of a non-ideal gas, explain whether the temperature increases or decreases.

Problem 8 (10 points)

Consider a system consisting two energy levels, zero energy and ε (ε>0) containing one particle at most.

(A) (2 pts) Demonstrate that the grand partition function for this system is

$$Z = 1 + e^{\beta\mu} + \exp(\beta(\mu - \varepsilon))$$

Where μ is a chemical potential, and $\beta = \frac{1}{k_B T}$

(B) (3 pts) Show that the average number of particles is given by $< n >= \frac{e^{\beta\mu} + \exp(\beta(\mu-\varepsilon))}{Z}$,

(C) (2 pts) Show that the probability of the system being in the state with energy ε is

$$\frac{\exp(\beta(\mu-\varepsilon))}{Z}$$

(D) (3 pts) Determine the average energy of the system.
Describe the high and low temperature limits of the average energy.

Problem 9 (10 points)

Consider the interferometer illustrated in Figure 1. A beam of light that has a range of wavelength between λ_{min} and λ_{max} is incident on a 50/50 beam splitter thereby separating the initial beam along two paths, with equal intensity along both paths. The split beams reflect back at mirrors 1 and 2 and propagate to the detector. The path length between the beam splitter and mirror 2 is L, whereas the path length to mirror 1 is L+x. Mirror 1 can be moved as a function of x, but Mirror 2 is fixed.

(a) (4 pts) The position of the mirror is fixed at $x = x_A$. All wavelengths emanating from the light source have the same intensity. However, the detector measures the intensity of the light to vary as a function of wavelength. There is one maximum at λ_1 and one minimum at λ_2. The relation between the wavelength and intensity is shown in Figure 2. Derive the relationship between x_A and λ_1 in terms of the difference in integer number of wavelengths m (m= 1, 2, 3,,,).

(b) (3 pts) The intensity at λ_2 in Figure 2 is the minimum intensity, find x_A by using λ_1 and λ_2.

(c) (3 pts) Mirror 1 is moved to x = 0. What is the measured intensity at different wavelengths?

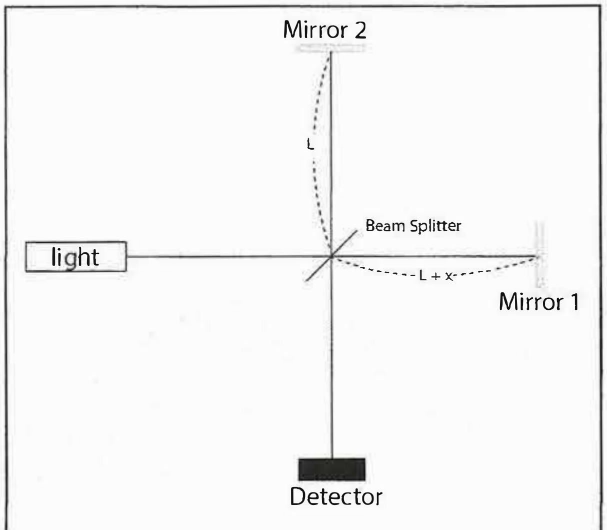

Figure 1

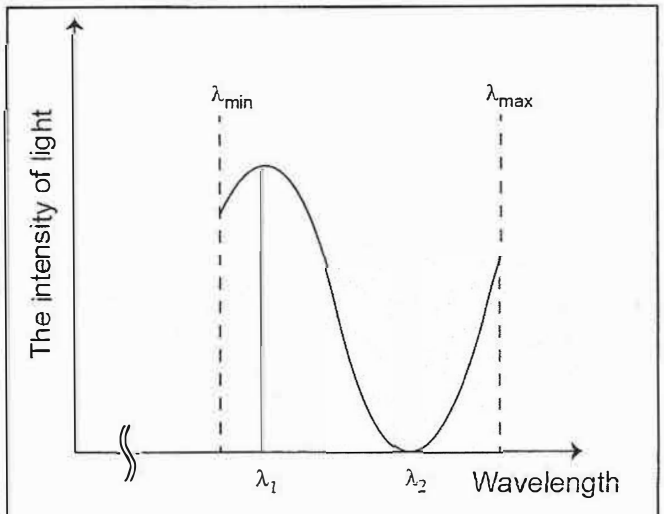

Figure 2

Ph.D. QUALIFYING EXAMINATION
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PART IV

MONDAY, May 6, 2019
13:30 — 15:30

ROOM 245 PHYSICS RESEARCH BUILDING

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Quantum Problems:

10. **(10 points)** Consider the case of the single particle wave function $\Psi(x)$ in one dimension.

a. **(4 points)** Consider a square well in one dimension where the potential $V(x) = 0$ from $-r < x < r$ and $V(x) = V > 0$ elsewhere. Consider a particle that is bound, with energy $E < V$ in this square well, and derive the full parity even solution $[\Psi(-x) = \Psi(x)]$, both inside and outside the well.

b. **(6 points)** In the limit that $V \to \infty$ in part **a.**, consider the state with the lowest non-zero energy. Define the uncertainty of the position of the particle as $\Delta x = \sqrt{|\langle \Delta x^2 \rangle|} = \sqrt{|\langle (x - \langle x \rangle|^2}$, and the uncertainty in the momentum as $\Delta p = \sqrt{|\langle \Delta p^2 \rangle|}$. The expectation can be evaluated as,

$$\langle \mathcal{O} \rangle = \int dx \;\; \Psi^*(x) \mathcal{O} \Psi(x). \tag{1}$$

Assuming that $\langle x \rangle = \langle p \rangle = 0$, calculate Δx and Δp for the case of the infinite well. Also calculate the product $\Delta x \Delta p$. Does a particle that has been confined within an enclosure obey Heisenberg's uncertainty principle?

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{2}$$

11. **(10 points)** Consider the case of the hydrogen atom. In the limit that the mass of the proton can be considered to be infinitely larger than the electron, the Schrödinger equation can be expressed as,

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) - \frac{e^2}{r}\psi(\vec{r}) = E\psi(\vec{r}). \tag{3}$$

In the equation above, $r = |\vec{r}|$ is the radial coordinate , E is the energy of the state, and e is the charge of the electron.

a. (3 points) Separate the wave-function and the Schrödinger equation into its radial and angular parts, i.e., write $\psi(\vec{r}) = R(r)Y^l_m(\theta, \phi)$, and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\hat{L}^2(\theta,\phi)}{r^2} \tag{4}$$

In the equation above, $\hbar\hat{L}$ is the angular momentum operator. Assuming $\hat{L}^2Y^l_m(\theta,\phi) = l(l+1)$, derive the equation for the radial portion only. Taking the asymptotic limit $r \to \infty$ of this equation, consider the case where the electron is bound. What is the form of $R(r)$ as $r \to \infty$, denoted as $R_\infty(r)$? What are the constraints on the energy E?

b. (3 points) Use the result of **a.** above to express the full radial equation as $R_\infty W(r)$, where $R_\infty W(r)$ is finite as $r \to 0$ or ∞. Derive the equation for $W(r)$ and solve using a power series expansion (Frobenius method). Assume $l = 0$, i.e., only consider the zero angular momentum solution.

c. (2 points) For the case of $l = 0$, derive a relation between the energy of the bound state and the mass and charge of the electron which fulfills the condition that $R_\infty W(r)$ is finite as $r \to \infty$

d. (2 points) The ground state energy for the case of $l = 0$ for the hydrogen atom is -13.6eV. Reformulate the Schrödinger equation for the case of positronium a bound state of an electron and its anti-particle the positron, which has the same mass as the electron but the opposite charge. What is the energy of the ground state of positronium also at $l = 0$?

12. **(10 points)** The angular momentum operators $\vec{J} \equiv [J_x; J_y, J_z]$ obey the following commutation relations,

$$[J_x, J_y] = i\hbar J_z \ ; \ [J_y, J_z] = i\hbar J_x \ ; \ [J_z, J_x] = i\hbar J_y. \tag{5}$$

a. (3 points) Show that it is possible to have states that are combined eigenstates $|j, m\rangle$ of $J^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

b. (5 points) What are the allowed eigenvalues of J^2 and J_z? Also prove that the angular momentum quantum number j can only assume an integer or a half-integer value.

c. (2 points) In the core particle model of nuclei, one models the atomic nucleus as a solitary nucleon orbiting a core made up of several other nucleons. For the case that the core has an angular momentum of $J = \sqrt{6}\hbar$ and a nucleon has an angular momentum of $\sqrt{3}\hbar/2$, what are the $|j, m\rangle$ states of the nucleus?