Ph.D. QUALIFYING EXAMINATION
DEPARTMENT OF PHYSICS AND ASTRONOMY
WAYNE STATE UNIVERSITY

PART I

Friday, May 7, 2021
10:00 — 12:00

ROOM 245 PHYSICS RESEARCH BUILDING

INSTRUCTIONS: This examination consists of three problems each worth 10 points. Use a separate booklet for each problem. Write the following information on the front cover of each booklet:

1. your special ID number that you received from Delores Cowen,
2. the problem number (*i.e.* Problem 7).

Please make sure your answers are dark and legible.

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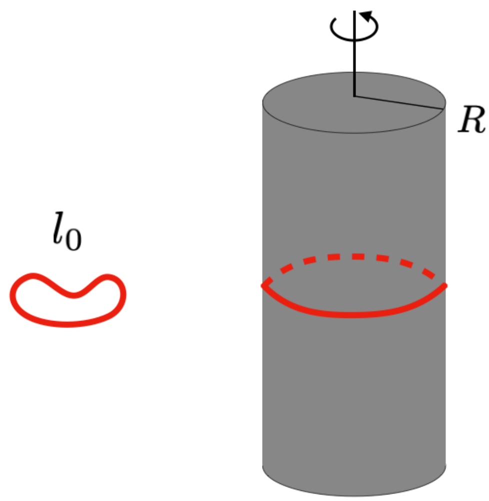
Problem 1 (10 points)

A rubber band with mass m has spring constant k and natural length l_0. Now, it is wrapped around the body of a massless cylinder with radius R so as to make a full turn. Here $2\pi R > l_0$. The rubber band is uniformly stretched.

(a) (4 points) Compute the magnitude and direction of the force per unit length which the rubber band exerts on the cylinder.

(b) (3 points) Now the cylinder starts to rotate along its z-axis with a constant angular acceleration, $\frac{d\vec{\omega}}{dt} = \vec{a}_0$. Here we assume the friction coefficient is large enough that the rubber band does not slip. Compute the torque on the cylinder to maintain this angular acceleration.

(c) (3 points) Compute the critical angular velocity ω_0 at which the cylinder does not feel the force from the rubber band anymore.



Problem 2 (10 points)

A spherical pendulum consists of a point mass m tied by a massless rod of length l to a fixed point, so that it is constrained to move on a spherical surface as shown in the figure below.

(a) (3 points) Write out the Lagrangian and derive the equation of motion for the system.

(b) (4 points) When the pendulum moves in a circle it will have angular velocity ω_0 and a constant angle θ_0 with the vertical. Based on the equation of motion show that

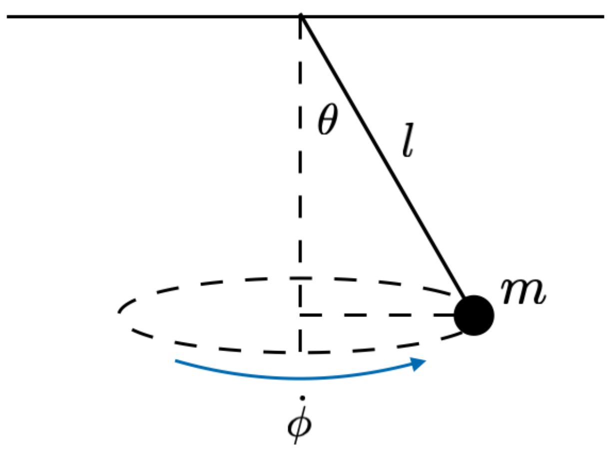
$$\dot{\phi}\sin^2\theta = \text{constant} = \omega_0 \sin^2\theta_0 \tag{1}$$

and the polar angle θ satisfies the following equation,

$$\ddot{\theta} - \omega_0^2 \sin^4\theta_0 \frac{\cos\theta}{\sin^3\theta} + \omega_0^2 \cos\theta_0 \sin\theta = 0. \tag{2}$$

(c) (3 points) The particle in the circular orbit as in part (a) above receives an impulse perpendicular to its velocity, resulting in an orbit which oscillates between polar angles θ_1 (highest) and θ_2 (lowest). At these two polar angles, $\dot{\theta} = 0$. Use the relation $\ddot{\theta} = d\dot{\theta}^2/(2d\theta)$ to integrate Eq. (2) and derive the following equation that the polar angles θ_1 and θ_2 satisfy,

$$\frac{1}{2}\omega_0^2 \sin^4\theta_0 \left(\frac{1}{\sin^2\theta_1} - \frac{1}{\sin^2\theta_2}\right) - \omega_0^2 \cos\theta_0(\cos\theta_1 - \cos\theta_2) = 0. \tag{3}$$



Problem 3 (10 points)

A particle of mass m is projected from infinity with a velocity V_0 in a manner such that it would pass a distance b from a fixed center O of inverse-square repulsive force (magnitude k/r^2, where k is a constant) if it were not deflected. The potential is $V(r) = k/r$. Show that

(a) (4 points) the closest distance R to the center point O is

$$R = \frac{k}{mV_0^2} + \sqrt{\frac{k^2}{m^2V_0^4} + b^2} \tag{4}$$

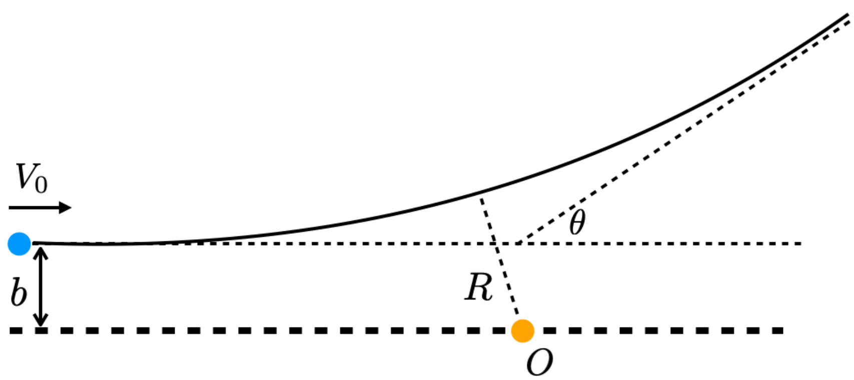
(b) (4 points) the angular of deflection θ is

$$\theta = 2\text{arctan}\left(\frac{k}{mV_0^2 b}\right). \tag{5}$$

(c) (2 points) Using the results from part (b), calculate the differential scattering cross section $d\sigma/d\Omega$ for a homogeneous beam of particles scattered by this potential is

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{4m^2V_0^4}\frac{1}{\sin^4(\theta/2)}. \tag{6}$$

[Useful information: The differential cross section is $d\sigma = 2\pi b db$.]



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PART II

Friday, May 7, 2021
13:30 — 15:30

ROOM 245 PHYSICS RESEARCH BUILDING

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Problem 4 (10 points)

A light plane wave enters a glass plane at a perpendicular angle to the surface. For this problem you can consider the index of refraction of air to be 1. For the glass, $n = \sqrt{\epsilon/\epsilon_0}$ and both media have $\mu = \mu_0$. The wave vectors of the incident, transmitted, and reflected waves are $\mathbf{k}$, $\mathbf{k}_T$, and $\mathbf{k}_R$, respectively.

(a) (1 point) Write the relations between the wave vectors $\mathbf{k}, \mathbf{k}_T, \mathbf{k}_R$ and the radiation frequency ω, and wave velocities v and v_T.

(b) (1 point) Using Maxwell's equation, derive for a linearly polarized plane wave in any linear medium the vector relations between $\mathbf{k}, \mathbf{E}, \mathbf{B}$ in a medium, including any proportionality constant.

(c) (4 points) Using boundary conditions at the dielectric surface, derive equations relating the incoming electric field E, and the reflected and transmitted fields E_R, E_T. Do the same for the magnetic field H. Solve for E_R as a function of E, n.

(d) (1 point) Derive the transmission and reflection efficiencies.

(e) (3 points) The incoming wave has intensity I (the dimension is Watts per square meter). Derive the radiation pressure on the interface.

Problem 5 (10 points)

Consider two equal pointlike charges of charge $+q$ located in the $x-$axis at $(b, 0)$ and $(-b, 0)$. A conducting grounded sphere of radius $R < b$ is placed at the origin.

(a) (3 points) Determine the location and charge of the two image charges.

(b) (2 points) It is now desired to have zero force acting on either charge. Write the equation for zero force acting on either charge.

(c) (3 points) Assume now that $R \ll b$ and solve perturbatively for R (consider only first non zero term).

(d) (2 points) Using the next term in the perturbation, how precise is the solution you found in (c)?

Problem 6 (10 points)

Three equal very long wires are located at $-d$, 0, and d along the x-axis . They are parallel to the y-axis and each carries a current I in the positive y-direction. Each has mass m and length L.

(a) (4 points) Find the locations where the magnetic field is zero.

(b) (3 points) The central wire is displaced from 0 to x_0, $x_0 \ll d$, at $t = 0$, along the x-axis, and is free to move. The other two wires are held fixed. Calculate the equation of motion of the wire under the force of the two other wires.

(c) (3 points) The central wire is displaced from 0 to z_0, $z_0 \ll d$, at $t = 0$, along the z-axis (that is, off-plane), and is free to move. The other two wires are held fixed. Calculate the equation of motion of the wire under the force of the two other wires.

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PART III

Monday, May 10, 2021
10:00 — 12:00

ROOM 245 PHYSICS RESEARCH BUILDING

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Problem 7 (10 points)

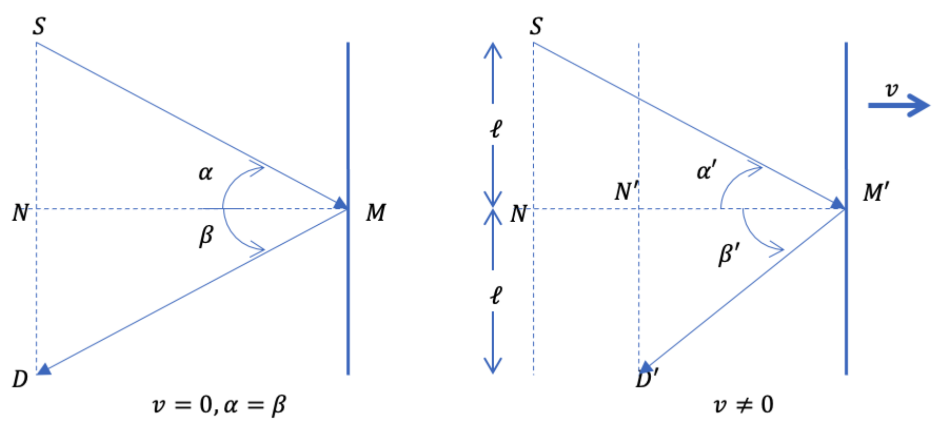
An experimental configuration, set up in a train, consists of a light source (S), a light detector (D) and a mirror (M). The line SD of length $2l$, joining the source and the detector, is parallel to the surface of the mirror. The line MN is normal to the mirror as shown. Two observers, O and O', are measuring the angles of incidence (α) and of reflection (β) in the experiment. Observer O is inside the train at rest with respect to the experimental configuration, while observer O' is standing outside the train at rest on the railway platform. When the train is at rest, both O and O' measure the angles α and β to be equal.

(a) (1 point) What is the relationship between angles α and β, as measured by O when the train is moving to the right with a constant speed v?

(b) (6 points) Show that the relationship between angles α' and β', as measured by O' when the train is moving to the right with a constant speed v, is

$$\cos\beta' = \frac{\cos\alpha'\left(1+\frac{v^2}{c^2}\right) - 2\left(\frac{v}{c}\right)}{\left(1+\frac{v^2}{c^2}\right) - 2\left(\frac{v}{c}\right)\cos\alpha'}. \tag{7}$$

(c) (3 points) With the train moving to the right with a constant speed v, for what value of v/c would O' measure the angle of incidence α' and the angle of reflection β' to be equal to $\pi/4$ and $\pi/2$, respectively?



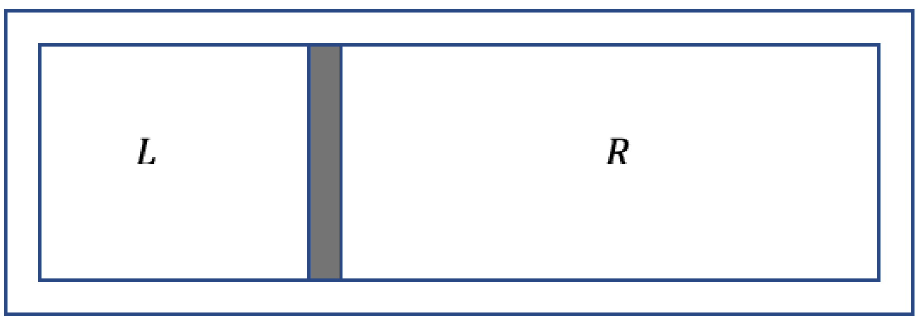
Problem 8 (10 points)

A thermally insulated cylinder is broken into two compartments, left (L) and right (R), by a gas-tight frictionless sliding piston. The piston has small thermal conductivity so that heat can be transferred between the two compartments without transferring any gas. Each compartment contains N moles of an ideal monoatomic gas. Initially (at $t = 0$), the absolute temperature of the gas in the left compartment is T_0 and of the gas in the right compartment is $3T_0$. The system is always in mechanical equilibrium (that is, the piston has no oscillations). Eventually (at $t = \infty$) the system will reach thermal equilibrium after sufficient heat transfers in irreversible manner from hotter compartment to cooler compartment via piston.

(a) (3 points) What is the ratio of volume of left (L) compartment to that of right (R) compartment at $t = 0$ and at $t = \infty$? What are the temperatures of the two compartments at $t = \infty$?

(b) (4 points) What is the total change in entropy of the entire system during this irreversible process between $t = 0$ and $t = \infty$?

(c) (3 points) How much useful work be done by the system if the transfer of heat from one compartment to the other could have been done by a reversible process?



Problem 9 (10 points)

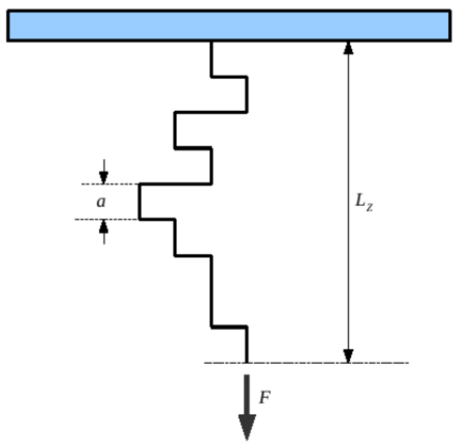
Consider a chain with $N \gg 1$ massless links of length a that can be oriented in three directions relative to the link above: left, right, or down, as shown below. Suppose that the upper end of the chain is fixed, and a constant force F is applied to the lower end of the chain, and the system is in a thermodynamic equilibrium at temperature T.

(a) (3 points) What is the mean end-to-end vertical extension, $\langle L_z \rangle$, of the chain?

(b) (3 points) Show that in the limit $Fa \ll kT$ the system satisfies Hook's law. Find the corresponding "spring constant".

(c) (2 points) Find the entropy of the system to the first order in $Fa/(kT) \ll 1$.

(d) (2 points) Estimate the fluctuations in the mean vertical extension, $\langle (\Delta L_z)^2 \rangle = \langle (L_z - \langle L_z \rangle)^2 \rangle$.



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PART IV

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13:30 — 15:30

ROOM 245 PHYSICS RESEARCH BUILDING

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Problem 10 (10 points)

A particle, of mass m, is free to move in the three-dimensional annular region between two impenetrable concentric spheres of radii a and $3a$. For this spherically symmetric situation, the Schrodinger equation is separable in the spherical coordinates. The angular part of the complete wave function is the standard spherical harmonic, $Y_{lm}(\theta, \phi)$, which depends on the orbital angular quantum number l and magnetic quantum number m.

(a) (2 points) Show that the radial part of the Schrodinger equation can be written in a form that resembles the one-dimensional wave equation in coordinate r.

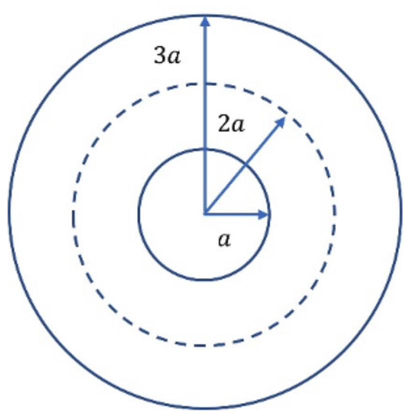
(b) (2 points) What are the boundary conditions on the wave function in three regions – the region outside the outer sphere, the region inside the inner sphere and the annular region?

(c) (3 points) In the annular region, for $l = 0$, show that the energy eigenvalues of the particle are of the form $E_n = n^2 E_1$ with $n = 1, 2, 3, \ldots$. Determine the value of E_1. Now imagine that the annular region is separated into two compartments by a fictitious concentric spherical surface of radius $2a$.

(d) (3 points) What is the ratio of probabilities of finding the particle in the outer compartment (enclosed by the surfaces of spheres of radius $3a$ and $2a$) and of finding the particle in the inner compartment (enclosed by surfaces of spheres of radius $2a$ and a) when it is in the lowest energy state?

[Useful information: $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$]

[Userful information: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\phi^2} = -l(l+1)Y_{lm}$]



Problem 11 (10 points)

A particle of the mass m is confined in a square well potential:

$$U(x) = \begin{cases} -U_0, & |x| < a \\ 0 & |x| > a \end{cases}, \tag{8}$$

where $ma^2U_0/\hbar^2 \ll 1$.

(a) (3 points) Write the Schrodinger equation for the wave function inside and outside of the well and the boundary conditions.

(b) (4 points) Prove that there exist only one bound state and find the corresponding energy

(c) (3 points) Calculate $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$. Check the uncertainty relation.

Problem 12 (10 points)

Consider a particle in 3-dimensional spherical harmonic oscillator potential $U(r) = m\omega^2 r^2/2$.

(a) (4 points) For the first four lowest energy states write down the corresponding wave functions in Cartesian coordinates.

(b) (3 points) Rewrite the corresponding wave functions in spherical coordinates, and express the angular part in terms of spherical harmonics.

(c) (3 points) If the total angular momentum and the projection of the angular momentum on the z-axis are measured in those states, what are the possible outcomes? What are the corresponding probabilities?

The wave functions for lowest energy state 1d quantum harmonic oscillator and lowest order spherical harmonics are provided below:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp[-m\omega x^2/(2\hbar)] \tag{9}$$

$$\psi_1(x) = 2m\omega x \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp[-m\omega x^2/(2\hbar)] \tag{10}$$

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}} \tag{11}$$

$$Y_{1,0} = i\sqrt{\frac{3}{4\pi}} \cos\theta \tag{12}$$

$$Y_{1,\pm1} = \mp i\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi} \tag{13}$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(1 - 3\cos^2\theta) \tag{14}$$

$$Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\, e^{\pm i\phi} \tag{15}$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{\pm i2\phi} \tag{16}$$