THH & the description of

\[ \text{THH}(F_p) \rightarrow \text{due to Bökstedt} \]

(unpublished), also done by Breen purely algebraically.

Still it seems that \( \text{THH}(F_p) \) as a ring one needs some topological input, e.g. Steenrod operations.

See Franjou - Lannes - Schwartz.

For a reference, Blomberg - Cohen - Schlichtkrull, compute \( \text{THH}(F_p) \) using \( F_p \) as a Thom spectrum.
See also notes by Krause-Nikolaus for a discussion. (on THH).

Topological cyclic homology appeared in Bökstedt-Hsiang-Madsen
\[ K(R) \to TC(R) \]

Theorem of Dundas-Goodwillie-McCarthy if \( R \) a ring, \( I \subset R \) nilpotent
have a cartesian square
\[ \begin{array}{ccc}
K(R) & \to & TC(R) \\
\downarrow & & \downarrow \\
K(R/I) & \to & TC(R/I)
\end{array} \]
Applications by Hesselholt, Madsen to compute $K$-theory of lots of rings.

→ See Madsen's survey "Algebraic $K$-theory & traces" as cyclotomic spectra

$$\text{TC}(R) = \text{Hom}(\mathbb{T}, \text{THH}(R))$$

Initially one didn't have $\text{CycSp}$ as a homotopy theory (instead $\text{TC}$ defined more explicitly).

Mentioned by Kaledin, 2010.
The $A$-homotopy theory $\text{CycSp}$ was constructed by Blumberg-Mandell, and the formula above is correct.

Also described by Barnick-Glasman, Ayala-Mazel-Gee-Rozenblyum.

Nikolaus-Scholze: all descriptions of cyclotomic spectra have redundancy in bounded-below case.

They give a description of cyclotomic spectra:

$$S' \overset{C \times} \longrightarrow X \overset{C} \longrightarrow X / C'$$

$$S' / C'$$
The description of $\text{THH}(R)$, $R$ a commutative ring from their paper.

Antieau - Nikolaus: Give a description of cyclotomic spectra in terms of the invariant $\text{TR} \rightarrow$ "topological Cartier modules."

Main theorem (Bhatt - Morrow - Scholze). $R$ smooth algebra / perfect field $k$.

Then there is a filtration on the

\[ \text{TP}(R) = \text{THH}(R)^{tS^1} \]

whose associated graded is
\((2\text{-periodic})\) crystalline cohomology of \(R\).

\[E_\infty \quad R = \mathbb{F}_p.\]

\[\text{THH}(\mathbb{F}_p)_\alpha = \mathbb{F}_p [\alpha], \quad |\alpha| = 2.\]

\[\text{TP}(\mathbb{F}_p)_\alpha = \mathbb{Z}_p \left[ x^{\frac{1}{2}} \right], \quad |x| = 2.\]

Crystalline cohom of \(\mathbb{F}_p\) is \(\mathbb{Z}_p\).

\(\text{Rmk}\) \(R\) any \(\mathbb{F}_p\)-algebra

\(\text{TP}(R)\) is a module over \(\text{TP}(\mathbb{F}_p)\)

\[\text{TP}(R) \otimes_\mathbb{F}_p \mathbb{F}_p \cong \text{H}^1(R/\mathbb{F}_p).\]
Constructing this filtration is somewhat subtle (Antieau-Nikolaus gave a simpler construction); BMS really want to input mixed char
nings (e.g. 24).

How to construct filtration on $\text{FP}(\mathbb{R})$?

1) Derive functors so can apply to large $\text{FP-alg}$. (derive differential forms, $\text{dR coh}$).

2) Construct the filtration directly for large $\text{FP-alg}$ ("regular semiperfect") (Postnikov filtration).
3) Define on smooth $\text{FP}_n$-algs by faithfully flat descent.

Derived functors ("nonabelian").

$\text{Symi}_i : \text{Vect}_k \rightarrow \text{Vect}_k$.

Admits a derived functor

$L \text{Symi}_i : D(k)_{>0} \rightarrow D(k)_{>0}$.

Explicitly, $p_i \rightarrow$ Dold-Kan into a simplicial $k$-vector space $p_i'$.

$\Rightarrow$ Consider $\text{Symi}_i p_i'$ as a new simplicial
k-vector space \( \rightarrow \) apply \( \mathbb{D}K \)
to a chain \( \varepsilon \). This is \( \mathbb{L}Sym^i \).

\[ \text{Example (Quillen cotangent complex).} \]
\[ \text{\( k \) a base field.} \]
\[ \text{Each comm \( k \)-alg \( R \) \( \rightarrow \) } \mathcal{L} \mathbb{R} \mathbb{H} \mathbb{L} / k \]/

Kähler differentials.

The "derived functor" is the cotangent complex \( \mathcal{L} \mathbb{R} \mathbb{H} \mathbb{L} / k \).

Construction: \( R \) a ring.) choose
a simplicial \( k \)-alg \( P, u / \)
\( P, \rightarrow R \) \( q \)-iso.
Then \( I_{R/k} = \left| \text{SL}_p / k \right| \)

geometric realization

More generally \( R \) can be a SCR.

Key property: \( H_0(L_{R/k}) = \text{SL}_R / k \).

Can be higher homology \( \sim \)

does not happen if \( R/k \) smooth.

Consider \( \text{SCR}_k \) \( \sim \) homotopy theory of simplicial comm rings

Poly\( k \) \( \subseteq \text{SCR}_k \)
Fact: If $\mathcal{C}$ is any $\infty$-category with sifted colimits,

$$\text{Fun}((\text{poly}_{k\mathbb{Z}}), \mathcal{C}) = \text{Fun}'((\text{CR}_k), \mathcal{C})$$

commutes with sifted colimits.

$$\mathcal{E} = \mathcal{D}(k),$$

$F$: $\mathcal{S}L/\mathbb{L}$ on poly nomial rings

\text{(Analog of this for derived symmetric exterior universal property for derived category)}
In general, it's hard to make this explicit on rings (which are not polynomial).

Ex) Quillen cotangent $\Omega_x$

$\text{Lie}_k = \text{SL}_R/k$

if $R$ smooth.

Another example: derived de Rham cohomology.

Recall ($k$ a base field)

$R/k$ alg (smooth).
Consider $(S_\infty/k_0 \otimes \mathbb{L}_{k_0/k}) \mapsto \text{commutative deg over } k$.

Smooth alg/k \quad \rightarrow \quad E_{\infty}-\text{alg over } k

R \quad \rightarrow \quad (S_\infty^* \otimes k_0)\mathbb{L}_k\).

Let's try to derive this construction (Illusie).

Restrict polynomial rings & then resolve (simplicial) vgs by polynomial rings.

\text{Dr}_k : S\mathbb{L}_k \rightarrow E_{\infty}-\text{alg over } k
"derived de Rham cohomology"

Ex) If $R$ is a poly ring, it's usual de Rham complex is general based on some resolution.

Theorem (Bhatt): $k$ perfect, char $p$, $\text{DR}_{R/K} \cong (\text{DR}_{R/K})^d$ if $R$ smooth, (Cartier isomorphism conjugate filtration).

Not true in char. 0!
Also a theory of derived crystalline cohomology agrees at ord. crystalline coh on smooth algs (k-alg).

Prop k base fieldiy R/k alg.

Claim is that \( HH(R/k) \) has a (converged) descending filtration whose grid

\[
\Delta L_{R/k} \llbracket \frac{1}{i} \rrbracket
\]

- consider \( L_{R/k} \) is in \( D(R) \)
- \( \Lambda^i \) is ith exterior in \( R \)-modules (duals)
Consider

\[ \text{HH}(L/k) : 
\text{SCR}_k \rightarrow \Omega(k). \]

Observe that this counts ul/sifted colimits.

Everything is determined by polynomial rings.

In fact, \( \text{HH}(L/k) \) is completely determined by value on polynomial rings.

If \( P \) is a polynomial ring,

\[ \text{HKR theorem}: \quad \text{HH}_{*}(P/k) \cong S^*_P/k. \]

\[ (\text{true for smooth algebras}). \]
Take the Postnikov filtration
\[ F^i \HH(p/k) = \tau_{\geq i} \HH(p/k) \]
\[ \gr^i = \left( \bigwedge^i \mathcal{S}_p/k \right)[i] \]

Proved prop if $p$ is polynomial
now extend formally (Kan extension)
to get the statement in general. \( \square \)

For this to be useful need rings for which $Lp/k$ known
Prop: \( k \) perfect, char. \( p \),

\( R/k \) is a perfect ring

meaning Frobenius: \( R \to R \) is an iso.

Then \( LR/k = 0 \).

Ex) \( \mathbb{F}_p \left[ x^{1/p^\infty} \right] \) is a perfect ring.

Proof: \( \deg 0 \) \( \mathbb{Z}_k(k) \)

\[ dx = 0 \quad \forall x \in R, \quad x = y^p \]

\[ dx = d(y^p) = py^{p-1}dy = 0 \text{ in char. } p. \]
Cor: \( R/k \) perfect \( \Rightarrow \)
\[
HH(R/k)_{\ast} = R. \quad (\text{in deg } 0)
\]

Pf: Use HKR filtration from previous prop. \( LR/k = 0. \quad \Box \)

Cor: \( \text{THH}(R)_{\ast} = R[\alpha] \).

\( R \) perfect

Describe \( \text{THH}(\mathbb{F}_p) \)
\[
_{\ast}^{!} \mathcal{H}_S^1 \approx \mathbb{Z}_p[x_3, \alpha \overline{\gamma}] \xrightarrow{\times \alpha = 0} \]

\( |\alpha| = 2 \)
\( |x_3| = -2 \quad (x \in H^2(\mathbb{F}_p)) \)
Nikolaus-Scholze $\Rightarrow$ $\text{THH}(F_p)$ is a cyclotomic spectrum

$\text{THH}(F_p) \approx T_{p^0} \left( \mathbb{Q}_p \right)$

as $\mathbb{E}_8$-thing