Theorem (Bhatt–Morava–Scholze):
\[ \mathbb{A}/k \] smooth, \( k \) perf. char. \( p > 0 \).

Then \( TP(\mathbb{A}) := THH(\mathbb{A}) \) has a filtration whose \( gr \approx 2 \)-periodic
\[ R[p_cups(\text{Spec}(R))] \]
integral version of de Rham
\[ coh_{\mathbb{A}} \approx \varinjlim \tilde{\mathbb{A}}/W(k) \]
whenever \( \tilde{\mathbb{A}} \) is a lift of
\[ \mathbb{A} \to W(k) \]
Supposed to be analogous to
the motivic filtration on algebraic
\( K \)-theory.
Ingredients in the proof of this theory

Last time talked about machinery of derived functors & derived de Rham cohomology, cotangent complex

\[ \Omega^1_{\mathcal{M}/k} \]

Prop. \( R/k \) any alg then there is a filtration on \( \text{HH}(R/k) \) whose associated gr is \( \bigoplus_{i} (L_{\mathcal{M}R/k})^{[i]} \)

\[ \text{PF: Postnikov filtration for } R \text{ smoothly derived functors everywhere.} \]
Example: 
\[ \text{HH}(R/k)_\times = R \] if \( R \) perfect \( F_p \)-alg. 

(k perfect field). char \( p \).

Cor: if \( R \) is a perfect \( F_p \)-alg.,
\[ \text{THH}(R) = R [\sigma], \quad |\sigma| = 2. \]

(b/c \( \text{THH}(RG) \cong \text{HH}(R/F_p) = R \)).

Theorem (BMS):
The construction \( R \to \text{THH}(R) \)
satisfies faithfully flat descent
as does \( R \to \text{TP}(R) \)
\[ \text{TC}(R) = \text{THH}(R) \]
\[ \ldots \]
More precisely, if $R \to S$, then

$$TP(R) \cong \text{Tot} \left( TP(S) \right) \cong TP(S \otimes R) \cong \ldots$$

Rink: In étale case, Weibel- Gelbg, McCarthy-Minasian

Idea: (Bhatt) First prove this for $L \otimes_{Fp}$. Here use cofiber sequences for $L \otimes_{Fp}$

(iF $A \to B \to C$ are maps of rings

cofiber seq $L_{B/A} \otimes_{B} C \to L_{C/A} \to L_{C/B}$

& then bootstrap to $\wedge L \otimes_{Fp}$
& show that this is a sheaf for flat topology.
& use HKR filtration $\Rightarrow$ $\text{HH}^*(/\text{FP})$
 & is also sheaf for flat topology.

$\& \text{THH}(R)/\sigma = \text{HH}(R/\text{FP})$

$\Rightarrow \text{THH}(R)$ a sheaf.
$\Rightarrow \text{TP}(R)$ a sheaf (connectivity argument)

A smooth $\text{FP}$-alg.
Idea is to use sheaf property to understand $\text{TP}(R)$. 
Construction \[ S = R_{perf} \]
\[ = \lim_{x \to x^p} R \]

Ex) \[ R = \mathbb{F}_p[x] \]
\[ R_{perf} = \mathbb{F}_p[x^{1/p}] = \bigcup_n \mathbb{F}_p[x^{1/p^n}] \]

Fact: \( R \) smooth \( \mathbb{F}_p \)-alg
\[ R \to R_{perf} - S \text{ is faithfully flat.} \]

To describe \( TP(R) \), it suffices to describe
\[ TP(S), \ TP(S \otimes \mathbb{F}_p S), \ldots \]
Idea is that these are easier (even degrees).
First case: $TP(S)$  $S$ perfect

\[ TP(S)_* = W(S)[x^{37}] , \, |x| = -2. \]

Prop: $TP(S)_* = W(S) P_{-1}, \, |o| = 2$

Run the $S^1$-Tate spectral sequence.
Identify the extension.

Filtration on $TP(S)$ is the Postnikov filtration: (crystalline coh of $S$
$\rightarrow W(S)$).

Unfortunately, this is not sufficient
$S \otimes_S S \otimes_S S \otimes_S ...$
$S \otimes S$ is not perfect.

Example: $R = \mathbb{F}_p[x, y]$ $S = \mathbb{F}_p[x^{1/p^\infty}]$.

$S \otimes_R S = \mathbb{F}_p\left[\frac{x^{1/p^\infty} \times x^{1/p^\infty}}{x_1 = x_2}\right]

This is no longer a perfect ring!

$(x_1^{1/p} - x_2^{1/p})^p = 0$

$S \otimes_R S$ is a "regular semi-perfect" ring

but the quotient of a perfect ring

by a regular sequence.

More generally, all tensor products $S \otimes_S \cdots \otimes_S$
are regular semi-perfect.

Needs to understand

1) $TP(T)_+$, $T$ regular semi-perfect

2) (derived) crystalline cohomology of $T$.

Main thm: (up to a mild completion) derivative crystalline cohomology of $T$ is a discrete ring, $TP(T)_+$ = $D$-periodic derived crystalline cohomology.
The BMS filtration on
\[ f_i \text{TP}(R) = \text{Tot}(\tau_{s2i} \text{TP}(S) \triangleright \tau_{s2i} \text{TP}(S)) \]
\[ \equiv \cdots \]

This gives a filtration on TP(R) & the fact that its associated is crystalline coh. follows from the analogous assertion on TP(S[r]).

How do we work w/ regular semi-perfect rings T?

Claim is that TP ... are all in even degree → this all boils down \( L_T / P_p \).
Prop 1: \( T \) is regular semi-perfect, \( L_{T/\mathfrak{p}} \) is suspension of a f.g. projective \( T \)-module, (in degree \( 1 \)).

Cor. \( T \) regular semi-perfect, \( HH(T/\mathfrak{p}), HH(T) \) are in even degrees.

Pf.: Consequence of HKR filtration \( HH(T/\mathfrak{p}) \) is filtered by

\[ g^i = \left( \wedge^i L_{T/\mathfrak{p}} \right) \mathfrak{F}^i \]
This also implies $THH(T)_h$ in even degrees.
$TP(T)_x$ is even.

Want to say explicitly what $TP(T)_x$ is, & relate to denied crystalline cohomology.

**Theorem:** $T$ regular semi-perfect ring then $L_{cryst}(T)$ (dened crystalline cohom.)

is a discrete ring in degree 0 & $A_{cryst}(T)$. Moreover $TP_0(T)$

is the completion of $A_{cryst}(T)$ w.r.t.

Nygaard filtration

$\Rightarrow$ leads to BMS filtration.
What is $\text{Acrys}(T) \equiv (\pi_g^* F_p [x]_{160}/x)$?

**Construction of Acrys...**

\[ T_{\text{perf}} \rightarrow T \]

\[ \lim_{x \rightarrow x^f} T \]

\[ (F_p [x]_{160}/x) \rightarrow F_p [x]_{160}^{16} \]

Consider $W(T_{\text{perf}}) \rightarrow T$.

$\Theta$ is the universal pro-nilpotent (radically complete).
thickening of $T$.

if $S ightarrow T$ which is a nilpotent

thickening

$\rightarrow W^{perf}(\cdot)$

\textbf{Def.}

$\text{Acrps}(T) = \text{divided power envelope of}$

$\rightarrow W^{perf}(\cdot)$, add all

divided powers of elements of kernel of $\Theta$.

(e.g. $x/\frac{i}{i!}$) $\times \ker \Theta$.

compatible with divided powers on $P$.}
\( \text{Aray}_s(T) \) is the universal (pro) PD thickening of \( T \)

in the sense

\[
\text{nilpotent thickening with divided power on } K
\]

Consider

\[
\text{TP}(T) \cong \text{Aray}_s(T)
\]

e.g., in degree 0

\[
\text{TP}(T)_0 = \Pi_0 \text{THH}(T)
\]

\[
\Pi_0 \text{THH}(T) \cong T
\]
"quasi syntonic site"

\[ \mathbb{Z}_p \mathbb{G} \rightarrow \hat{\mathbb{Z}}_p \mathbb{G} \]

\[ \equiv \]

\[ \mathbb{R} \]

\[ S \]