An Introduction to Chromatic Homotopy Theory

Part IV: Morava E-Theory and the Stabilizer Group

Agnès Beaudry

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(1) Chromatic convergence

\[ X_{(p)} \simeq \text{holim} L_nX \]

where

\[ L_nX = L_{K(0) \vee \ldots \vee K(n)} X \simeq L_{E(n)} X \simeq L_{v_n^{-1}MU(p)} X \]

(2) The algebraic chromatic filtration

\[ \mathcal{C}_n^a(X) = \ker(\pi_* X \to \pi_* L_{n-1}X). \]

(3) Type n spectra: finite \( X \) s.t. \( K(n)_* X \neq 0 \) and \( K(n-1)_* X = 0 \)

(4) Type n spectra have a \( v_n \)-self map

(5) \( v_n \)-self maps give rise to periodic families in homotopy groups

(6) For \( X \) a type n spectrum with \( v_n \)-self map \( f \), let \( \text{Tel}(n) = f^{-1}X \) and

\[ L^f_nX = L_{\text{Tel}(0) \vee \ldots \vee \text{Tel}(n)} X \]

(7) The geometric chromatic filtration

\[ \mathcal{C}_n^g(X) = \ker(\pi_* X \to \pi_* L^f_{n-1}X). \]

(8) Telescope Conjecture: \( L^f_nX \simeq L_nX \) and, so, the two filtrations agree.
Part IV – Morava $E$-Theory and the Stabilizer Group

(1) The Chromatic Splitting Conjecture
(2) Adams-Novikov Spectral Sequence
(3) Morava $E$-Theory and the Morava Stabilizer Group
(4) $L_{K(n)}S^0$ as Homotopy Fixed Points
Why care about the Telescope Conjecture?

The Telescope Conjecture predicts an equivalence

\[ L_n^f X \overset{\sim}{\to} L_n X \]

- \( L_n^f \) has an intuitive description in terms of inverting \( v_n \)-self maps
- \( L_n \) has chromatic convergence.
- \( L_n \) is more computable than \( L_n^f \).
For any $X$, there is a homotopy pull-back:

\[
\begin{array}{ccc}
L_n X & \longrightarrow & L_{K(n)} X \\
\downarrow & & \downarrow \\
L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X
\end{array}
\]

called the **Chromatic Fracture Square**.

- $L_n X$ is obtained by gluing $L_{K(n)} X$ and $L_{n-1} X$ along $L_{n-1} L_{K(n)} X$
- $L_{n-1} L_{K(n)} X$ together with the maps
  
  \[
  L_{n-1} X \to L_{n-1} L_{K(n)} X \leftarrow L_{K(n)} X
  \]

  form gluing data.
- $L_{K(n)} X$ are the building blocks.

The Chromatic Splitting Conjecture is about the gluing data.
Weak Chromatic Splitting Conjecture (Hopkins)

For a finite spectrum $X$, with $X_p$ its $p$-completion, the map

$$L_{n-1}X_p \longrightarrow L_{n-1}L_{K(n)}X_p$$

is the inclusion of a wedge summand.

- $n = 1, \ p \geq 2$ Classical computations of Adams-Bousfield-Baird-Ravenel
- $n = 2, \ p > 5$ Hopkins based on Shimumura-Yabe
- $n = 2, \ p = 3$ Goerss-Henn-Mahowald
- $n = 2, \ p = 2$ Beaudry-Goerss-Henn
- $n > 2, \ p \geq 2$ Wide Open

Chromatic Convergence for $L_{K(n)}$

The Chromatic Splitting Conjecture implies that, for finite spectra $X$,

$$X_p \simeq \operatorname{holim}_n L_{K(n)}X$$
How do we study the chromatic layers \( L_n S^0 \) and \( L_{K(n)} S^0 \)?

... More fundamentally, how do we study \( S^0 \)?
Adams Novikov Spectral Sequence

Let \( \iota: S^0 \to MU \) be the unit and \( \mu: MU \wedge MU \to MU \) be the multiplication. The map \( \iota \) is a poor approximation of \( S^0 \). For example,

\[
\pi_* S^0 \to \pi_0 S^0 \cong \mathbb{Z} \hookrightarrow MU_*
\]

But ... we keep going and get an augmented cosimplicial spectrum

\[
\begin{array}{ccccccc}
S^0 & \to & MU & \leftarrow & MU \wedge MU & \leftarrow & \ldots \\
\iota \wedge 1 & & & & 1 \wedge \iota & & \\
\iota & \mu & & 1 & & \\
\end{array}
\]

and

\[
S^0 \cong L_{MU} S^0 \to \cong \text{Tot}(MU^*)
\]

This story gives rise to the *Adams-Novikov Spectral Sequence* (ANSS)

\[
\text{Ext}_{MU_* MU}^*(MU_*, MU_*) \Longrightarrow \pi_* S^0
\]

where the Ext here is the cohomology of a cobar complex

\[
0 \to MU_* \xrightarrow{\pi_*(\iota \wedge 1 - 1 \wedge \iota)} MU_* MU \to MU_* (MU \wedge MU) \to \ldots
\]
The Adams–Novikov spectral sequence for $p = 5$, $t - s \leq 240$, and $s \geq 2$.
Mahowald Uncertainty Principle

Any spectral sequence converging to the homotopy groups of sphere with an $E_2$-term that can be named using homological algebra will be infinitely far away from the actual answer.

Instead, we try to climb the chromatic ladder and study $L_n S^0$ and $L_{K(n)} S^0$. 
Fix $p$. Recall that $E(n)$ is the Johnson-Wilson spectrum and $L_n \simeq L_{E(n)}$, so

\[ S^0 \xrightarrow{\iota} E(n) \xrightarrow{\rightarrow} L_n S^0 \]

**Theorem.**

\[ L_n S^0 \cong \text{Tot} (E(n)^\bullet) \]

The $E(n)$-based ANSS

\[ \text{Ext}_{E(n)}^{*,*} (E(n)^*, E(n)^*) \rightarrow \pi_* L_n S^0 \]

The $E(n)$-ANSS has a horizontal vanishing line at some $E_r$-page.
Morava $E$-Theory

Fix $p$. Let

- $\mathbb{Z}_{p^n} = \mathbb{Z}_p(\zeta)$ for $\zeta$ a primitive $p^n - 1$ root of unity, so $\mathbb{Z}_{p^n}/p = \mathbb{F}_{p^n}$.
- $(E_n)_* = \mathbb{Z}_{p^n}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$ for $u_i \in E_0$, $u \in E_2$

Let $E_n$ be the Landweber theory associated to the composite

$$MU_* \rightarrow E(n)_* \cong \mathbb{Z}_p(v_1, \ldots, v_{n-1})[v_{n-1}^{\pm 1}] \rightarrow (E_n)_*$$

$$v_i \mapsto \begin{cases} u_i u^{1-p^i} & i < n \\ u^{1-p^n} & i = n \\ 0 & i > n \end{cases}$$

The spectrum $E_n$ is called Morava $E$-Theory. For $n = 1$, $E_1 = K_p$.

- $L_n \cong L_{E_n}$
- $E_n$ is a periodic version of $L_{K(n)} E(n)$
- $E_n$ is $K(n)$-local, so $L_{K(n)} E_n \cong E_n$
- However, $E_n \wedge E_n$ is not $K(n)$-local, so we declare:

$$(E_n)_* E_n := \pi_* L_{K(n)}(E_n \wedge E_n)$$
Since $E_n$ is $K(n)$-Local

\[ S^0 \xrightarrow{\iota} E_n \]

\[ \text{Theorem.} \]

\[ L_{K(n)}S^0 \xrightarrow{\simeq} \text{Tot} \left( L_{K(n)}(E_n^\bullet) \right) \]

The $K(n)$-local $E_n$-based ANSS

\[ \text{Ext}^*_{(E_n)^*}(E_n^*, (E_n)^*)_E \xrightarrow{\pi_*} \pi_*L_{K(n)}S^0 \]

The $K(n)$-$E_n$-ANSS has a horizontal vanishing line at some $E_r$-page.
Where does $E_n$ fit in our formal group law story?
A **strict isomorphism** of formal group laws is a homomorphism

\[ f : F \to G \quad f(x +_F y) = f(x) +_G f(y) \quad f(F(x, y)) = G(f(x), f(y)). \]

such that

\[ f(x) = x + \sum_{i \geq 1} r_i x^{i+1} \in xR_*[[x]]. \]

Form the groupoid \( \widehat{FGL}(R_\ast) \) of formal group laws with strict isomorphisms:

\[ \widehat{FGL} : \text{Graded Rings} \to \text{Groupoids} \]

A morphism in \( \widehat{FGL}(R_\ast) \) is a triple \( (F, G, f : F \xrightarrow{\cong \text{s.i.}} G) \).

**Note.** If \( F \) is a formal group law and \( f(x) \) is as above, then

\[ G(x, y) = f^{-1}(F(f(x), f(y))) \]

So, \( F(x, y) \) and \( f(x) \) completely determines \( f : F \to G \).

\[ MU_\ast MU = \pi_\ast(MU \wedge MU) \cong MU_\ast[b_1, b_2, \ldots] \]

and \( \phi : MU_\ast MU \to R_\ast \) classifies a morphism in \( \widehat{FGL}(R_\ast) \) where \( r_i = \phi(b_i) \).
Functors to Groupoids

- \((MU_*, MU_*MU)\) represents 
  \(\widehat{FGL}: \text{Graded Rings} \to \text{Groupoids}\)
- \((E(n)_*, E(n)_*E(n))\) represents (roughly) 
  \(\widehat{FGL}_{\leq n}: \text{Graded } p\text{-Local Rings} \to \text{Groupoids}\)
- \(((E_n)_*, (E_n)_*E_n)\) is related to a functor 
  \(\text{Def}: \text{Complete Local Rings} \to \text{Groupoids}\)

- \(K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]\). Let \(\phi: K(n)_* \to K(n)_0\) map \(v_n\) to 1 and
  \[\Gamma_n := \phi_*F_{K(n)} \in FGL(K(n)_0)\]
  where \(K(n)_0 = \mathbb{F}_p\).
- \((E_n)_* = \mathbb{Z}_p^n[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]\). Let \(\varphi: (E_n)_* \to (E_n)_0\) map \(u\) to 1 and
  \[F_n := \varphi_*F_{E_n} \in FGL((E_n)_0)\]
  where \((E_n)_0 = \mathbb{Z}_p^n[[u_1, \ldots, u_{n-1}]]\).
Let $B$ be a complete local ring with maximal ideal $m$ and let
\[ q : B \to B/m. \]

A deformation of $\Gamma_n$ over $B$ is a pair $(i, G)$ such that
\[ G \in FGL(B), \quad i : F_p^n \to B/m \]
\[ q_* G = i_* \Gamma_n, \quad \Gamma_n \cong i_* \Gamma_n \leftarrow q_* G \]

A $*$-isomorphism of deformations $(G, i)$ to $(G', i)$ is an isomorphism
\[ f : G \cong \rightarrow G' \]
such that $q_* f = \text{id}$:

\[
\begin{array}{ccc}
G & \xrightarrow{f} & G' \\
q_* \downarrow & & \downarrow q_* \\
i_* \Gamma_n = q_* G & \xrightarrow{q_* f} & q_* G' = i_* \Gamma_n
\end{array}
\]

Let $\operatorname{Def}_{\Gamma_n}(B)$ be the groupoid of deformations and $*$-isomorphisms. Note that
\[ \operatorname{Def}_{\Gamma_n}(B) = \bigsqcup_{i : F_p^n \to B/m} \operatorname{Def}_{\Gamma_n}(B)_i. \]
The Lubin-Tate Theorem

Provided that $\text{Def}_{\Gamma_n}(B)$ is not empty, there are bijections

- $\pi_0 \text{Def}_{\Gamma_n}(B)_i \cong m \times (n-1)$. This corresponds to choices for the coefficients of $x^p^k$, $k = 1, \ldots, n - 1$ in $[p]_G(x)$.
- $\pi_1 \text{Def}_{\Gamma_n}(B)_i = \{1\}$. That is, $\exists! \star$-iso between $(G, i) \cong_\star (G', i)$.

Furthermore, the formal group law $F_n$ over

$$(E_n)_0 = \mathbb{Z}_{p^n}[u_1, \ldots, u_{n-1}], \quad m_E = (p, u_1, \ldots, u_{n-1})$$

represents $\pi_0 \text{Def}_{\Gamma_n}(B)$:

$$\text{Hom}^c_{\text{Rings}}(E_0, B) \cong \pi_0 \text{Def}_{\Gamma}(B), \quad (\phi: (E_n)_0 \to B) \mapsto \phi_\star F_n \cong_\star G$$

\[
\begin{array}{ccc}
(E_n)_0 & \xrightarrow{\phi} & B \\
\downarrow & & \downarrow q \\
k & \xrightarrow{i} & B/m
\end{array}
\quad
\begin{array}{ccc}
F_n & \xrightarrow{\sim} & \phi_\star F_n & \xrightarrow{\cong_\star} & G \\
\downarrow & & \downarrow q_* & & \downarrow q_* \\
\Gamma_n & \xrightarrow{\sim} & i_\star \Gamma_n & \xrightarrow{=} & q_* G
\end{array}
\]
The Morava Stabilizer Group

Let
\[ G_n = \{ (f, i), \ i \in \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p), \ f : i_\ast \Gamma_n \xrightarrow{\cong} \Gamma_n \} \supseteq \text{Aut}(\Gamma_n) \]

with product
\[ (g, j)(f, i) = (g \circ j \ast f, j \circ i) \]
\[ (ji)_\ast \Gamma_n \xrightarrow{j_\ast f} j_\ast \Gamma_n \xrightarrow{g} \Gamma_n \]

The group \( G_n \) acts on \( (E_n)_0 \). Let \( \alpha = (f, i) \) and \( g(x) \in (E_n)_0[[x]] \) be s.t.
\[ g(x) \equiv f(x) \mod m_E \]

The pair \( (G, i) \) where
\[ G(x, y) = g^{-1}F_n(g(x), g(y)) \]
is a point in \( \text{Def}_{\Gamma_n}((E_n)_0) \). There is a homomorphism \( \phi^\alpha : (E_n)_0 \to (E_n)_0 \) s.t.
\[ \phi^\alpha F_n \xrightarrow{\cong} G \xrightarrow{g} F_n \]
\[ i_\ast \Gamma_n \xrightarrow{=} i_\ast \Gamma_n \xrightarrow{f} \Gamma_n \]
Higher Adams Operations

\( G_n \) acts via \( \phi^\alpha \) on \((E_n)_0\). This extends to an action on \((E_n)_\ast\) by

\[
\phi^\alpha (u) = (\phi^\alpha)'(0)^{-1} \cdot u
\]

**Example.** If \( n = 1 \),

- \( E_1 = K_p, \ (E_1)_\ast = \mathbb{Z}_p[u^{\pm 1}] \) where \( u = \beta \), the Bott class.
- \( F_1(x, y) = x + y + xy \) over \( \mathbb{Z}_p \)
- \( \Gamma_1(x, y) = x + y + xy \) over \( \mathbb{F}_p \)
- \( \mathbb{G}_1 \cong \mathbb{Z}_p^\times \) with action on \((E_1)_\ast\) the opposite of the Adams operations.

The group \( G_n \) is the group of higher Adams operations.

Goerss-Hopkins-Miller Theorem

The group \( G_n \) acts on the spectrum \( E_n \) by maps of \( \mathcal{E}_\infty \) ring spectra.
Group Cohomology

For a discrete group $G$, the cobar complex

$$(C^*(G, M), \delta)$$

is obtained from the cosimplicial abelian group

$$\text{Hom}(G^\bullet, M) = \left( M \xrightarrow{} \text{Hom}(G, M) \xrightarrow{} \text{Hom}(G^2, M) \xrightarrow{} \cdots \right)$$

by letting $\delta$ be the alternating sums of face maps, and group cohomology

$$H^*(G, M) = H^*(C^*(G, M), \delta).$$

Galois Extensions

A $G$-Galois extensions of rings is a ring homomorphism $S \rightarrow R$ such that

- $R$ has an action of $G$ by ring homomorphisms.
- $S \xrightarrow{\cong} R^G$
- $R \otimes_S R \cong \text{Hom}(G, R)$. 
Co-Operations for $E$-Theory

There are isomorphisms

$$(E_n) \ast E_n = \pi \ast L_{K(n)}(E_n \wedge E_n) \cong \text{Hom}_c(\mathbb{G}_n, (E_n) \ast)$$

More generally, there is an isomorphism of cosimplicial sets

$$\pi \ast L_{K(n)}(E_n) \cong \text{Hom}_c(\mathbb{G}_n, (E_n) \ast)$$

The Galois Extension

This lifts in spectra to equivalences

$$L_{K(n)}(E_n) \cong F_c((\mathbb{G}_n)^{\bullet}, E_n)$$

so that

$$L_{K(n)}S^0 \cong \text{Tot}(L_{K(n)}(E_n^{\bullet+1})) \cong \text{Tot}(F_c((\mathbb{G}_n)^{\bullet}, E_n) =: E_n^h \mathbb{G}_n.$$  

The $E_n$-based $K(n)$-local ANSS is a homotopy fixed point spectral sequence

$$H_c^\ast(\mathbb{G}_n, (E_n) \ast) \Longrightarrow \pi \ast L_{K(n)}S^0$$

Together, these things imply that

$$L_{K(n)}S^0 \rightarrow E_n^h \mathbb{G}_n$$

is a $\mathbb{G}_n$-Galois extension of ring spectra.
The $K(1)$-Local Sphere at $p = 2$

The $E_2$ (top) and $E_\infty$ (bottom) pages of the spectral sequence

$$H_c^*(\mathbb{G}_1, (E_1)_*) \Rightarrow \pi_* L_{K(1)} S^0$$

Here, a $\square$ denotes a copy of $\mathbb{Z}_2$, a $\bullet$ denotes a copy of $\mathbb{Z}/2$, a $\bigcirc$ a copy of $\mathbb{Z}/4$ and so on. Dashed lines denote exotic multiplications by 2.
Thank you!